Dynamic Analysis Tool Development for Advanced Geometry Wind Turbine Blades

By

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## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>List of Figures</td>
<td>vi</td>
</tr>
<tr>
<td></td>
<td>List of Tables</td>
<td>viii</td>
</tr>
<tr>
<td></td>
<td>Nomenclature</td>
<td>ix</td>
</tr>
<tr>
<td></td>
<td>Abstract</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td><strong>1 Introduction</strong></td>
<td>2</td>
</tr>
<tr>
<td>1.1</td>
<td>Current Status of Wind Energy</td>
<td>2</td>
</tr>
<tr>
<td>1.2</td>
<td>Advanced Wind Turbine Rotors</td>
<td>3</td>
</tr>
<tr>
<td>1.3</td>
<td>Wind Turbine Analysis</td>
<td>5</td>
</tr>
<tr>
<td>1.4</td>
<td>Research Objectives and Motivation</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td><strong>2 Blade Finite Element Modeling</strong></td>
<td>10</td>
</tr>
<tr>
<td>2.1</td>
<td>Introduction</td>
<td>10</td>
</tr>
<tr>
<td>2.2</td>
<td>Method</td>
<td>11</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Coordinate Systems and Transformations</td>
<td>11</td>
</tr>
<tr>
<td>2.2.2</td>
<td>Kinematics</td>
<td>17</td>
</tr>
<tr>
<td>2.2.3</td>
<td>Lagrange’s Equation</td>
<td>20</td>
</tr>
<tr>
<td>2.2.4</td>
<td>Steady State Equations and Axial Force</td>
<td>25</td>
</tr>
<tr>
<td>2.2.5</td>
<td>Oscillation about Steady State</td>
<td>28</td>
</tr>
<tr>
<td>2.2.6</td>
<td>Beam Element Matrices</td>
<td>29</td>
</tr>
<tr>
<td>2.2.7</td>
<td>Solution for Blade Frequencies and Mode Shapes</td>
<td>30</td>
</tr>
<tr>
<td>2.3</td>
<td>Finite Element Method Verification Results</td>
<td>32</td>
</tr>
<tr>
<td>2.3.1</td>
<td>Curved Beam Deflection Verification</td>
<td>32</td>
</tr>
<tr>
<td>2.3.2</td>
<td>Non-Rotating Curved Beam Modes Verification</td>
<td>35</td>
</tr>
<tr>
<td>2.3.3</td>
<td>Rotating Tapered Beam Verification</td>
<td>35</td>
</tr>
<tr>
<td>2.3.4</td>
<td>Rotating Beams from Leung and Fung 1988 Verification</td>
<td>38</td>
</tr>
<tr>
<td>2.3.5</td>
<td>Rotating Curved Beam Verification</td>
<td>45</td>
</tr>
<tr>
<td>2.4</td>
<td>Conclusions and Recommendations</td>
<td>47</td>
</tr>
</tbody>
</table>
3 Dynamic Analysis of Curved Wind Turbine Blades

3.1 Introduction

3.2 Coordinate Systems and Transformations

3.2.1 Local Undeflected Axis

3.2.2 Local Deflected Coordinate System

3.2.3 Local Aerodynamics Coordinate System

3.3 Kinematics

3.3.1 Positions and Displacements

3.3.2 Velocities

3.3.3 Accelerations

3.4 Kinetics

3.4.1 Generalized Inertia Forces

3.4.2 Generalized Active Forces

3.5 Blade Loads

3.5.1 Blade Root Loads

3.5.2 Blade Gage Moment Outputs

3.6 Verification with FAST and Adams

3.7 Validation with Field Test Data

3.8 Conclusions

3.8.1 Verification with Adams

3.8.2 Validation with Field Test Data

3.9 Recommendations

4 Design Studies

4.1 Introduction

4.2 Baseline Studies

4.2.1 Method

4.2.2 Results

4.2.3 Conclusions and Discussion

4.3 Scaling to Larger rotors

4.3.1 Method

4.3.2 Results

4.3.3 Conclusions and Recommendations

Bibliography

A Blade Finite Element Matrices

A.1 Mass Matrix

A.2 Elastic Stiffness Matrix

A.3 Gyroscopic Matrix

A.4 Spin Stiffness Matrix

A.5 Axial Force Stiffness Matrix

A.6 Axial Reduction Matrix

B CurveFEM Program Structure
C  Summary of FAST Modifications

--v--
List of Figures

1.1 Swept blade concept. .................................................. 4
1.2 Analysis flow diagram. .................................................. 6

2.1 First transformation of element ....................................... 13
2.2 Second transformation of element with built-in structural twist ........................................ 14
2.3 Beam element coordinate system ................................... 18
2.4 Beam element axial force (after Leung [1]) ....................... 27
2.5 Nodal degrees of freedom ............................................. 30
2.6 Curved beam for deflection verification (after Kosmatka [2]) ........................................... 32
2.7 Curve beam deflection and slope for $F_x = 100$ lb ................ 33
2.8 Curve beam deflection and slope for $F_y = 100$ lb ................. 34
2.9 Curved beam for natural response (after Kosmatka [2]) ........... 35
2.10 Rotating tapered beam (all dimensions in inches) ................ 36
2.11 Tapered beam verification results ................................... 37
2.12 Horizontal Cantilever (after Leung [1]) .......................... 38
2.13 Horizontal Cantilever Verification Results ....................... 39
2.14 Inclined Cantilever (after Leung [1]) ............................. 40
2.15 Inclined Cantilever Verification Results ......................... 41
2.16 L-beam (after Leung [1]) ............................................ 42
2.17 L-beam with $\theta = 90^\circ$ Verification Results ................... 43
2.18 L-beam with $\theta = 30^\circ$ Verification Results .................... 44
2.19 Rotating curved beam verification results ....................... 46

3.1 First transformation of element ....................................... 50
3.2 Second transformation of element with built-in structural twist ........................................ 51
3.3 Undeflected blade element offsets ................................... 57
3.4 Sweep angle for swept blade ........................................ 78
3.5 Blade tip twist verification ........................................... 98
3.6 Blade tip deflection verification .................................... 99
3.7 Generator power verification ........................................ 100
3.8 Flap bending moment verification ................................... 101
3.9 Edge bending moment verification ................................. 102
3.10 Out-of-plane deflection for STAR7d first flap bending mode .......... 104
3.11 Torsional deflection for STAR7d first flap bending mode .......... 105
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.12</td>
<td>Generator power validation</td>
<td>107</td>
</tr>
<tr>
<td>3.13</td>
<td>Blade pitch validation</td>
<td>108</td>
</tr>
<tr>
<td>3.14</td>
<td>Edge bending validation</td>
<td>109</td>
</tr>
<tr>
<td>3.15</td>
<td>Flap bending validation</td>
<td>110</td>
</tr>
<tr>
<td>4.1</td>
<td>Parametric study of damage equivalent loads</td>
<td>118</td>
</tr>
<tr>
<td>4.2</td>
<td>Parametric study of annual energy production</td>
<td>119</td>
</tr>
<tr>
<td>4.3</td>
<td>Parametric study of blade tip deflection</td>
<td>120</td>
</tr>
<tr>
<td>4.4</td>
<td>Flap bending comparison for scaled design</td>
<td>127</td>
</tr>
</tbody>
</table>
List of Tables

2.1 XYZ Component Values ................................................. 15
2.2 Curved beam modes verification ................................. 35

3.1 CurveFAST Blade 1 Degrees of Freedom ......................... 53
3.2 CurveFAST Degrees of Freedom ................................ 56
3.3 Turbulent wind file parameters .................................. 96
3.4 Maximum percentage differences for the Adams/CurveFAST verification 97
3.5 Extreme Load Verification, Normalized Maximum Values .......... 103

4.1 Parametric Studies .................................................. 115
4.2 WindPACT model wind turbine parameters ..................... 122
4.3 WP1500 blade properties ........................................... 123
4.4 WP3000 blade properties ........................................... 124
4.5 Scaled model results ................................................ 126
Nomenclature

Roman

\( a \) acceleration vector
\( A \) sectional area
\( C \) matrix for Kane’s method
\( C_D \) drag coefficient
\( C_{DA} \) drag coefficient modified for sweep
\( C_L \) lift coefficient
\( C_{LA} \) lift coefficient modified for sweep
\( C_M \) moment coefficient
\( D \) damping matrix
\( EA \) extensional stiffness
\( EI \) bending stiffness
\( f \) vector in equation of motion for Kane’s method
\( F \) element axial force
\( F \) force vector
$F_r$ generalized active forces

$F^*_r$ generalized inertia forces

$g$ gravitational acceleration

$g'$ hub primed system unit vectors

$G$ gyroscopic matrix

$GJ$ torsional stiffness

$h$ undeflected blade element offset

$H$ angular momentum vector

$i$ coned system unit vectors

$I$ identity matrix

$\overline{I}$ inertia dyadic

$j$ pitched system unit vectors

$J'$ polar moment of inertia about centroidal axis

$K$ combined stiffness matrix

$K^*$ modified stiffness matrix

$K_e$ elastic stiffness matrix

$K_g$ axial force stiffness matrix

$K_\Omega$ spin stiffness matrix

$l$ direction cosine, element length

$L$ beam length
\( L_j \) local undeflected element system unit vectors

\( m \) inverse of the Wöhler (or S-N) curve, direction cosine

\( m \) element aerodynamic system unit vectors

\( M \) mass matrix

\( M \) moment

\( M \) moment vector

\( M^* \) modified mass matrix

\( n \) number of degrees of freedom, number of cycles for a particular stress level, direction cosine

\( n \) local deflected element system unit vectors

\( N \) allowable number of cycles for a particular stress level

\( N \) shape function matrix

\( q \) CurveFAST degree of freedom

\( q \) nodal displacement vector

\( Q \) apex of coned system

\( r \) radial position

\( r \) position vector

\( s \) distance along beam element

\( S \) coefficient in axial reduction equation

\( t_e \) element trailing edge system unit vectors
\( T \)  kinetic energy

\( T_{CH'} \)  hub (primed) to coned system transformation

\( T_{DE} \)  undeflected to deflected element system transformation

\( T_{EP} \)  pitched to undeflected element system transformation

\( T_{PC} \)  coned to pitched system transformation

\( u \)  deflection in \( x \)-direction

\( u \)  displacement vector

\( U \)  strain energy

\( v \)  deflection in \( y \)-direction

\( v \)  velocity vector

\( V \)  wind velocity

\( w \)  deflection in \( z \)-direction

\( x_r \)  eigenvector

\( y_r \)  real part of eigenvector \( x_r \)

\( z_r \)  imaginary part of eigenvector \( x_r \)

\( z \)  tower system unit vectors

**Greek**

\( \alpha \)  direction cosine

\( \alpha \)  angular velocity

\( \beta \)  pitch angle
ζ modal damping ratio
θ angle, twist
λₜ eigenvalue
λ transformation matrix
Λ angle between relative velocity and chord line
μ lineal density
ξ non-dimensional position along element
ρ mass density
Φ interpolation function
ψ eigenvector element
Ψ matrix of eigenvector elements
ω angular velocity vector
Ω rotational speed
Ω spinning matrix

**Subscripts**

0 value at inboard element node

a applied, aerodynamic

Aero aerodynamic

ave average

AxRed axial reduction
BL  blade 1

Damp  damping

e  element

Elastic  elastic

eq  Equivalent

E1  first edge mode

F1  first flap mode

F2  second flap mode

Grav  gravitational

l  value at outboard element node

r  partial velocity derivative

s  structural, steady state

T1  first torsion mode

Teet  teeter degree of freedom

u  ultimate

v  oscillatory component

Superscripts

BLTip  blade 1 tip node

E  inertial frame

H  hub body
$MI$ blade 1 element body

$SI$ blade 1 element center of mass

**Other Notation**

( ′ ) differentiation with respect to $s$

( ′ ) time derivative

( )$^T$ transpose

**Acronyms**

AEP annual energy production

BEM blade element momentum method

CFD computational fluid dynamics

COE cost of energy

DEL damage equivalent load

DOE United States Department of Energy

DOF degree of freedom

EISG Energy Innovations Small Grant

IEC International Electrotechnical Commission

LWST Low Wind Speed Technology program

NREL National Renewable Energy Laboratory

NTM normal turbulence model

RPS renewable portfolio standard
STAR  swept-tip adaptive rotor

WindPACT  wind partnerships for advanced component technology
Abstract

This dissertation describes work to develop a dynamic analysis code for swept wind turbine blades. Because of their aeroelastic behavior, swept blades offer the potential to increase energy capture and lower fatigue loads. This work was an outgrowth of United States Department of Energy contract on swept blades, where the author used the Adams™ dynamic software. The author based the new code on the National Renewable Energy Laboratory’s FAST code. The new code would allow for lower cost analysis and faster computation times for swept blades compared to Adams. The FAST revisions included the geometry and mode shapes required for the bending and twisting motion of the swept blade. The author also developed a finite-element program to determine mode shapes for the swept blade. The author verified the new code with Adams. The comparisons were favorable; however, the Adams model exhibited more twist. The differences may be attributed to differences in modeling approach. The author attempted to validate the code with field test data; however, uncertainties in the test wind speed and the turbine controller made comparison difficult. The author used the new code to perform preliminary designs of swept rotors for 1.5 MW and 3.0 MW wind turbines. The designs showed a 5% increase in annual energy production and a decrease in flap-bending fatigue over the baseline straight-blade designs.
Chapter 1

Introduction

1.1 Current Status of Wind Energy

At the beginning of the 21st century, wind energy has become the fastest-growing new energy source. Worldwide installation continues at an exponential rate [3]. Large corporations have recently entered the business, such as BP, General Electric, Shell, and Siemens. Wind energy in California was 1.5% of the gross system energy production in 2005 [4]. Under the Renewables Portfolio Standard (RPS), California has mandated that renewables will account for 20% of the energy production by 2010. The majority of this production is expected to come from wind energy.

The growth in the wind energy industry can be attributed to competitive cost of energy (COE) in comparison to traditional energy supplies. Wind energy is consistently cited as being on par with the cost of generation by natural gas. Tangible benefits of wind energy include zero fuel costs in addition to zero greenhouse gas and airborne pollutant emissions.

The dramatic decrease in COE since the 1980’s has coincided with the increase in size of wind turbines. In the Solano area, for instance, ninety 1980’s vintage Kenetech 100 kW machines have each been replaced by six GE 1.5 MW units. The diameter of the old 100 kW turbine is 17 m, while the new 1.5 MW turbine diameter is 77 m. The 1.5 MW turbine
benefits from an increase in available wind power with the square of the rotor diameter, and also from increased wind speeds at higher hub heights due to wind shear. Larger diameter rotors are also necessary for increasing the wind energy potential at low wind speed sites that are closer to major metropolitan load centers.

However, these added benefits of larger diameter rotors potentially come at increasing COE. With an increase in rotor diameter, if all dimensions remain proportional the mass will increase with the cube of the diameter due to the volume increase. Mass is directly related to cost. This relationship is commonly referred to as the “square cube law.” In addition to the mass increasing by the cube of the diameter, the stress on the blades due to gravity increases in proportion to the rotor diameter increase [5]. These two effects, mass and stress increase, require that blades must be proportionally lighter and produce lower stress in order for the COE to not increase with increasing rotor diameter.

With this engineering challenge to continue the reduction of COE, the U.S. Department of Energy (DOE) initiated the Low Wind Speed Technology (LWST) program to move research in the direction of capturing more wind energy in low wind speed areas that require larger rotor diameters, thus positively affecting the economics for these regions.

1.2 Advanced Wind Turbine Rotors

One thrust of the LWST program is in advanced rotor control concepts. The idea is to reduce loads for larger rotor diameters so that they can be made proportionally lighter. Because the rotor typically accounts for 20% of the capital costs of a turbine [6], an increase in the cost of the rotor for these advancements would be more than offset by increased energy capture.

One concept [7] is to incorporate flap/twist coupling into the rotor with off-axis fiber orientation into the blade construction. As the blade deflects in the flapwise direction, the tip twists toward feather and reduces the aerodynamic loading. A different physical
mechanism, leading to similar aerodynamic load reduction, is analyzed by Zuteck [8] and proposed earlier by Liebst [9]. This concept is to sweep the outboard-region rotor planform in the plane of rotation aft of the pitch axis. The loads generated at the tip then introduce a moment about the pitch axis. With sufficient blade torsional flexibility, the tip twists toward feather, thus reducing the loads. This concept is illustrated below in Figure 1.1. A wire-frame model of a turbine with swept blades is shown in the left hand corner of Figure 1.2

![Figure 1.1: Swept blade concept.](image)

In 2004 the Blade Division of Knight & Carver won an LWST contract administered through Sandia National Labs. Knight & Carver assembled a team to design, manufacture, and test a rotor based on the swept-blade concept. The rotor would be designed for atmospheric testing on a Zond Z-50 wind turbine with 750 kW rating. The project team called the concept the STAR (Sweep-Twist Adaptive Rotor). The author was responsible for the dynamic analysis of the prototype rotor. Preliminary results show the load reduction effects [10].

The design goal of the project was to increase annual energy capture of the baseline turbine by 5%-10%. The rotor swept area was increased by 25% to increase the below-rated energy capture by 25% for a straight-bladed rotor. For the Z-50 turbine, the rotor
radius was increased from 25 m to 28 m. With sweep and twist, it was expected that the increase in below-rated energy capture would be 15%-20% and that therefore the overall annual energy capture would increase by 5%-10%. The turbine power rating would not be increased; therefore, there would be no increase in above-rated energy capture.

1.3 Wind Turbine Analysis

The design and analysis of wind turbines is a well-established field with several texts on the subject, such as Burton, Sharpe, Jenkins, and Bossanyi [6], Gasch and Twele [5], and Manwell, McGowan, and Rogers [11]. Molenaar [12] extensively surveys the specific topic of dynamic analysis for wind turbines. The wind energy industry has matured to a level such that full-aeroelastic dynamic modeling is used to design and analyze wind turbines. Wind turbine dynamic modeling is a complicated interaction of turbine structural dynamics, rotor/wake aerodynamics, and atmospheric boundary-layer fluid dynamics. The primary use of dynamic modeling is design and loads evaluation to ensure proper sizing of components. Engineers perform the evaluation according to industry accepted standards, such as the International Electrotechnical Commission (IEC) [13] or Germanischer Lloyd [14].

Figure 1.2 shows a typical process to be followed for wind turbine dynamic analysis. The National Renewable Energy Laboratory (NREL) developed and maintains the tools [15] in this diagram for wind turbine analysis. In the process, the analyst develops input files of the turbine properties. The FAST [16] code uses for input the machine properties, such as dimensions, masses, and inertias. FAST is a wind turbine specific dynamic analysis code. Aerodynamic and geometric properties of the blade are needed as input for the AeroDyn [17] aerodynamics subroutines. Simulated wind environments are built using TurbSim [18] and IECWind [19] codes. IECWind produces gust-type simulations, whereas TurbSim produces full-field three-dimensional turbulence. FAST is then executed to build
input files for MSC/Adams™. Currently, FAST does not have the capability of analyzing rotors with blade sweep. Adams is a commercial code used for multi-body dynamic simulation in many industries, including the aerospace and automotive sectors. NREL has worked on interfacing to Adams for wind turbine analysis [20]. The Adams models include lumped-parameter representation of the rotor blades. During execution, Adams solves the equations of motion at each time step, and the AeroDyn subroutines are called to compute the aerodynamic forces. The output is time series data of requested parameters that can be processed as desired.

Previous work has primarily focused on straight wind turbine blades. Recent interest in blades with sweep have pushed the limits of wind turbine analysis tools. Currently the tool with the most modeling fidelity is the commercial code Adams. The author used Adams for modeling the Knight and Carver swept blade for the LWST program [10]. Liebst [9] performed an early analysis of a swept-blade wind turbine rotor; however, the blade
model had constant cross-section, an assumption that would not work with modern blade construction. Neither analysis is validated with test data.

While Adams is a sophisticated modeling tool, it has several drawbacks that will be discussed below. Several wind-turbine specific modeling tools have been developed for the industry, including NREL’s FAST code, Garrad Hassan’s Bladed code from the United Kingdom, and Stig Øye’s FLEX5 code from Denmark. These codes are recognized by Germanischer Lloyd for loads evaluation of wind turbines. The primary modeling difference between Adams and these other codes is that Adams models the blade with lumped parameters, whereas the other codes use prescribed mode shapes to model the blade.

For the LWST project, the Knight and Carver team obtained great confidence in the swept blade design concept through the use of Adams. However, during the analysis the author determined that continuing to use Adams would be a hindrance to the further development of the concept. At the time Adams was being used under an academic license, and it was expected that a commercial license would have to be obtained for further work. A recent quote for Adams was $32,000 with a 20% annual fee [21]. The other two codes, Bladed and FLEX5, had capabilities of modeling swept blades. Garrad Hassan quoted Bladed at 25,000 British pounds. The project team was unable to obtain a quote for FLEX5.

Another major problem with Adams was slow run times, with a typical 10-minute turbulence simulation taking 3.5 hours (using Dell Latitude D600 with Pentium M 1400 MHz processor and 512 MB RAM). NREL has mentioned the run-time problem in the literature [20]. The run times were exacerbated by rotor-torque simulation problems, which required shorter time steps and longer run-times for convergence. The Adams models also showed sensitivity to certain turbulent wind file random-seeds. These slow run times make for difficult rapid design evaluation and automated loads analysis.

The author decided that modifying an existing wind turbine code with public-domain source code would be the best option for further work on the swept blade. The only code in the public domain with Germanischer Lloyd certification is NREL’s FAST [22]; therefore,
the author chose this code for new development. Wilson, Freeman, Walker, and Harman [23] at Oregon State University originally developed FAST, which has recently been updated by NREL [24]. The code uses Kane’s Method [25] for derivation of the equations of motion. Both the blade and tower are modeled as flexible elements with prescribed mode shapes [26]. Bladed and FLEX5 use similar methods. Currently, FAST includes three blade mode shapes: first flap-bending, second flap-bending, and first edge-bending. Use of mode shapes reduces the modeling complexity compared to the lumped-parameter representation in Adams. In addition, developing the program specifically for wind turbine analysis adds to the computational efficiency. The author has found that a model with straight blades in FAST can take seven times longer with Adams.

1.4 Research Objectives and Motivation

The objectives for this doctoral thesis work were to increase the modeling capability of the FAST code for the analysis of swept rotor blades, and to study the feasibility of the swept-blade concept for turbines of greater size than the STAR rotor. As mentioned above, these advanced-geometry blades offer the possibility of extending the rotor diameter of an existing straight-blade design, allowing increased energy capture without exceeding the design load envelope.

The motivation behind increasing the modeling capability of FAST, besides being of educational interest to the author, was to lower the cost and to increase the computation speed of analyzing the swept rotor. Along with high cost, Adams simulation times for conventional wind turbine designs were seven times the run time of FAST. Similar concerns are raised by NREL [20]. An example design case is to run all of the operating conditions for a particular design with 10-minute turbulent wind simulations according to the IEC Standard [13]. There would be 11 different wind speeds if 2 m/s steps were taken between 4 m/s and 24 m/s. For each wind speed, six turbulent simulations each would have to
run according to the standard. The total for this design case is 66 simulations. With an anticipated FAST run time of 30 minutes, this design case would take 33 hours. With Adams at approximately seven times the run time of FAST, the design case would take over nine days. Increasing the capability of FAST, without increasing the simulation time substantially, shows great promise as a design optimization tool.

The modification of FAST for swept blade required solution of the blade mode shapes. The author developed a finite element code for this purpose, called CurveFEM that is described in Chapter 2. Chapter 3 describes the equation of motion for the modifications to FAST, called CurveFAST. The author verified the changes for the swept blade with Adams. The author also compared CurveFAST model runs with field test data from the STAR rotor. A prototype of the STAR rotor ran on a turbine in Tehachapi, California, in the winter of 2008.

The author also studied the feasibility of scaling the swept blade concept to larger wind turbine rotors. Current production wind turbines vary from 1.5 MW to 3 MW, with offshore prototypes up to 5 MW. Key geometric parameters of the blade were studied for optimum design, such as the shape of the blade sweep curve, the torsional flexibility of the blade, and the maximum amount of sweep at the blade. There are important design limitations due to transportation of the blades from the manufacturer to the wind plant. It may be that the concept would not be feasible for larger blades due to these constraints. With a modified FAST, these design studies could be automated.

With an increasing rotor diameter, and lighter blades as required by the economics discussed above, comes the potential for flutter stability problems as discussed by Lobitz [27]. The author studied the flutter boundary were studied for the STAR rotor.

Chapter 4 describes these design and flutter studies.
Chapter 2

Blade Finite Element Modeling

2.1 Introduction

NREL’s FAST program [26] models wind turbine blades and towers as flexible bodies. The blade response is a linear summation of the lower bending modes. Authors [26][28] commonly call this technique the normal mode summation (or superposition) method. Currently, FAST uses three bending modes: first flap, first edge, and second flap. The modes couple through built-in twist. NREL provides the Modes program [29] to generate these mode shapes for straight, rotating, pitched, and tapered blades. Modes uses a Rayleigh-Ritz method as described in Jonkman [26] to determine mode shapes.

For blade sweep to be incorporated into FAST, the program requires new mode shapes that allow for twist deflections to occur during transverse bending motion. The author decided to retain the mode summation method in FAST, and therefore investigated methods for determining mode shapes of rotating, tapered, twisted, and curved beams. The propeller and rotorcraft (helicopter) research community has generated work on curved blades. The primary goal of the research is to use sweep to delay the onset of drag divergence. Secondary goals of the rotorcraft community are to potentially reduce hub loads. Bielawa [30] and Kosmatka [2][31] have works related to high-speed propellers. Bauchau and Hong
Bir [33], Celi and Friedmann [34], Hodges [35][36], Rosen and Rand [37][38][39], and Tarzanin and Vlaminick [40] have works related to rotorcraft. Many of these works model non-isotropic properties that are important for very flexible composite rotor blades. Most of the researchers also include non-linear behavior, which is important for rotorcraft stability analysis. Although Adams allows non-linear motion, most of the capabilities described by these researchers is beyond that currently modeled in Adams for wind turbines.

The author found that Leung and Fung [1] produced the most applicable and best documented work for the current project, titled “Spinning Finite Elements.” Leung and Fung produced this work for general aerospace applications. Their model uses standard linear finite elements of constant properties with six degrees of freedom (DOF) at each node. The author adapted their work for tapered properties into the new program CurveFEM, as explained in Section 2.2 below. This section describes the coordinate systems and kinematics of the finite elements, and the use of Lagrange’s equation to build the equations of motion (EOM). The finite element matrices derive from this development. The EOMs then become an eigenvalue problem. The solution to the eigenvalue problem are the natural frequencies and mode shapes of the blade. Section 2.3 describes the verification of CurveFEM with several published examples.

2.2 Method

2.2.1 Coordinate Systems and Transformations

This section documents the coordinate systems used by CurveFEM, and are common to FAST. Some of the coordinate systems have the superscript $B1$ for blade one; there are similar coordinate systems for the remaining blades. The first coordinate system is the hub (primed) coordinate system ($g'^{B1}$). This system rotates with the hub, with the $g'_1^{B1}$ unit vector parallel to the hub rotation axis. The hub rotates at constant speed $\Omega$, which is the same assumption in Modes. The $g'_3^{B1}$ unit vector would align to the blade pitch axis given
zero coning. The next system is the coned $i^{B1}$ system related to the $g^{B1}$ by:

\[
\begin{bmatrix}
i_1^{B1} \\
i_2^{B1} \\
i_3^{B1}
\end{bmatrix} = [T_{CH}] 
\begin{bmatrix}
i_1^{B1} \\
i_2^{B1} \\
i_3^{B1}
\end{bmatrix} \tag{2.1}
\]

with the transformation matrix:

\[
[T_{CH}] = 
\begin{bmatrix}
\cos[PreCone(1)] & 0 & -\sin[PreCone(1)] \\
0 & 1 & 0 \\
\sin[PreCone(1)] & 0 & \cos[PreCone(1)]
\end{bmatrix}
\]

where $PreCone(1)$ is the hub pre-coning angle for blade 1, which is positive downwind. Currently there are no provisions for coning in CurveFEM; therefore, the $T_{CH}$ matrix is equal to the identity matrix $[I]$. Coning could be added later to CurveFEM. The transformation to the blade pitched coordinate system ($j^{B1}$) is:

\[
\begin{bmatrix}
j_1^{B1} \\
j_2^{B1} \\
j_3^{B1}
\end{bmatrix} = [T_{PC}] 
\begin{bmatrix}
i_1^{B1} \\
i_2^{B1} \\
i_3^{B1}
\end{bmatrix} \tag{2.2}
\]

with the transformation matrix:

\[
[T_{PC}] = 
\begin{bmatrix}
\cos[BlPitch(1)] & -\sin[BlPitch(1)] & 0 \\
\sin[BlPitch(1)] & \cos[BlPitch(1)] & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

where $BlPitch(1)$ is the blade 1 pitch. The pitch angle is relative to the chord line at zero aerodynamic twist and is positive toward feather (leading edge upwind). Currently there is no provision for pitch (can be added later) in CurveFEM; therefore the transformation
matrix $T_{PC}$ is equal to the identity matrix $[I]$.

The next coordinate system is the blade element system aligned with the local undeflected-axis. It has the unit vectors $\mathbf{L}_i^{BI}$ ($i = 1, 2, 3$). The transformation from the pitch system is:

$$
\begin{bmatrix}
\mathbf{L}_1^{BI}(r) \\
\mathbf{L}_2^{BI}(r) \\
\mathbf{L}_3^{BI}(r)
\end{bmatrix} = [\mathbf{T}_{EP}(r)]
\begin{bmatrix}
\mathbf{j}_1^{BI} \\
\mathbf{j}_2^{BI} \\
\mathbf{j}_3^{BI}
\end{bmatrix}
$$

(2.3)

The transformation development follows a similar method to Rao [28] p. 328, for finite elements, but with a coordinate system change to match the FAST elements. The first stage involves a transformation matrix $[\lambda_1]$ between the pitched coordinates $XYZ$ and the coordinates $\bar{x} \bar{y} \bar{z}$ by assuming the $\bar{y}$ axis to be parallel to the $YZ$ plane (Fig. 2.1):

$$
\begin{bmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{bmatrix} = [\lambda_1]
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
$$

The next step is between the local coordinates $xyz$ (principal axes) and the coordinates $\bar{x} \bar{y} \bar{z}$
as (Fig. 2.2):
\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
= [\lambda_2]
\begin{bmatrix}
  \bar{x} \\
  \bar{y} \\
  \bar{z}
\end{bmatrix}
\]

Figure 2.2: Second transformation of element with built-in structural twist

The desired transformation between the \(xyz\) and the \(XYZ\) system is therefore:

\[
[T_{EP}(r)] = [\lambda_2][\lambda_1]
\]

where:
\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
= [T_{EP}(r)]
\begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix}
\]
From Fig. 2.1, the direction cosines of the longitudinal axis ($\bar{z}$ or $z$) are:

$$l_{oz} = l_{oz} = \frac{X_j - X_i}{l}$$
$$m_{oz} = m_{oz} = \frac{Y_j - Y_i}{l}$$
$$n_{oz} = n_{oz} = \frac{Z_j - Z_i}{l}$$

where the length of the element is:

$$l = \left\{ (X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2 \right\}^{1/2}$$

As in the FAST2ADAMS.f90 preprocessor, the interior element nodes coincide with the “analysis” nodes, which are interpolated from the properties in the blade input file. The last node uses the values at the tip. Table 2.1 shows values of the $XYZ$ components and their FAST variable names.

<table>
<thead>
<tr>
<th>Component</th>
<th>Inner Elements</th>
<th>Tip Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>PrecrvRef(i)</td>
<td>PrecrvRef(TipNode)</td>
</tr>
<tr>
<td>$Y$</td>
<td>PreswpRef(i)</td>
<td>PreswpRef(TipNode)</td>
</tr>
<tr>
<td>$Z$</td>
<td>BlFract(i) ×</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BldFlexL +</td>
<td></td>
</tr>
<tr>
<td></td>
<td>HubRad</td>
<td></td>
</tr>
</tbody>
</table>

Because the unit vector $\bar{j}_2$ (parallel to the $\bar{y}$ axis) is normal to both the unit vectors $\bar{j}_1$ (parallel to the $X$ axis) and $\bar{j}_3$ (parallel to the $\bar{z}$ axis) the following vector analysis relation holds:

$$\bar{j}_2 = \frac{\bar{j}_3 \times \bar{j}_1}{\|\bar{j}_3 \times \bar{j}_1\|} = \frac{1}{d} \begin{vmatrix} j_1 & j_2 & j_3 \\ l_{oz} & m_{oz} & n_{oz} \end{vmatrix} = \frac{1}{d} \left( j_2 n_{oz} - j_3 m_{oz} \right)$$

where:

$$d = \left( m_{oz}^2 + n_{oz}^2 \right)^{1/2}$$
Therefore, the direction cosines of the $\bar{y}$ axis with respect to the $XYZ$ system are:

\[ l_{o\bar{y}} = 0, \quad m_{o\bar{y}} = \frac{n_{oz}}{d}, \quad n_{o\bar{y}} = -\frac{m_{oz}}{d} \]

Because the $\bar{x}$ axis (unit vector $\bar{j}_1$) is orthonormal to the $\bar{y}$ axis ($\bar{j}_2$) and the $\bar{z}$ axis ($\bar{j}_3$), $\bar{j}_1$ is:

\[ \bar{j}_1 = \bar{j}_2 \times \bar{j}_3 = \begin{vmatrix} j_1 & j_2 & j_3 \\ l_{o\bar{y}} & m_{o\bar{y}} & n_{o\bar{y}} \\ l_{oz} & m_{oz} & n_{oz} \end{vmatrix} \]

\[ = \frac{1}{d} \begin{bmatrix} j_1(m_{oz}^2 + n_{oz}^2) + j_2(-l_{oz}m_{oz}) + j_3(-l_{oz}n_{oz}) \end{bmatrix} \]

The direction cosines for the $\bar{x}$ axis are therefore:

\[ l_{o\bar{x}} = \frac{m_{oz}^2 + n_{oz}^2}{d}, \quad m_{o\bar{x}} = -\frac{l_{oz}m_{oz}}{d}, \quad n_{o\bar{x}} = -\frac{l_{oz}n_{oz}}{d} \]

Therefore, the $[\lambda_1]$ matrix is:

\[ [\lambda_1] = \begin{bmatrix} l_{o\bar{x}} & m_{o\bar{x}} & n_{o\bar{x}} \\ l_{o\bar{y}} & m_{o\bar{y}} & n_{o\bar{y}} \\ l_{oz} & m_{oz} & n_{oz} \end{bmatrix} = \begin{bmatrix} (m_{oz}^2 + n_{oz}^2)/d & -(l_{oz}m_{oz})/d & -(l_{oz}n_{oz})/d \\ 0 & n_{oz}/d & -m_{oz}/d \\ l_{oz} & m_{oz} & n_{oz} \end{bmatrix} \]

The principal axes of the element ($xyz$ axes) makes an angle $\theta_s$ (built-in structural twist)
about the negative \( \bar{z} \) axis. This direction is such that a positive \( \theta_s \) results in a lower angle of attack. The transformation between the systems is:

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta_s & -\sin \theta_s & 0 \\
  \sin \theta_s & \cos \theta_s & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  \bar{x} \\
  \bar{y} \\
  \bar{z}
\end{bmatrix}
\]

so that:

\[
[\lambda_2] =
\begin{bmatrix}
  \cos \theta_s & -\sin \theta_s & 0 \\
  \sin \theta_s & \cos \theta_s & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

Note that this procedure breaks down when the \( z \) (or \( \bar{z} \)) axis is aligned to the \( X \) axis. In this case, \( m_{ox} = n_{ox} = 0 \) and therefore \( d = 0 \). However, this situation is not realizable for a horizontal-axis wind turbine because it implies that two stations are at the same radial position.

### 2.2.2 Kinematics

This section draws on the work of Leung and Fung [1], mentioned by the author in the introduction. They develop the element matrices for rotating, linear, constant property, space-frame elements with six DOF at each node. The author changed their coordinate systems to match the alignment in FAST, in addition to adding tapered element properties.

The beam element is located in space with three sets of orthogonal axes:

1. \( xyz \) along the local principal axes of the beam (with unit vectors \( \mathbf{L}_j \))

2. \( XYZ \) that is the pitched blade system for no pre-cone or pitch (with unit vectors \( \mathbf{j}_i \)) with \( X \) parallel to the spinning axis

3. \( \bar{X} \bar{Y} \bar{Z} \), an inertial frame with \( \bar{X} \) parallel to \( X \)
Figure 2.3 shows the first two coordinate systems.

The undeformed element is at rest relative to frame $XYZ$ and the position vector for any point on the element is:

$$\{r_0\} = \{r_g\} + \xi(\{r_h\} - \{r_g\}), \quad 0 \leq \xi \leq 1$$

or:

$$\{r_0\} = \{r_g\} + s\{s\}, \quad 0 \leq s \leq l$$ (2.4)

where $\{r_0\}$, $\{r_g\}$, and $\{r_h\}$ are respectively the position vectors of the point and the ends of the beam with respect to frame $XYZ$, $\{s\}$ is the unit vector along the beam axis with $\{s\} = \{r_{gh}\}/|r_{gh}|$ and $\{r_{gh}\} = \{r_h\} - \{r_g\}$.

A spinning matrix $[\Omega]$ used in the ensuing analysis for the element is:

$$[\Omega] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$ (2.5)
The absolute position vector for any point on the beam is:

\[ \{r\} = [X, Y, Z]^T \]

The absolute velocity vector with respect to the inertial frame is:

\[ \{v\} = \{\dot{r}\} + [\Omega] \{r\} \]

The absolute acceleration vector with respect to the inertial frame is:

\[ \{a\} = \{\ddot{r}\} + 2[\Omega] \{\dot{r}\} + \Omega \Omega \{r\} \]

The displacement vector \( \{u\} \) in the principal beam axis with reference to the local coordinates \( xyz \) is:

\[ \{u\} = [u, v, w]^T \]

where \( u, v, w \) are displacements along the \( xyz \) axes respectively. The displacement vector \( \{\bar{u}\} \) in the principal beam axis with reference to the moving coordinates \( XYZ \) is:

\[ \{\bar{u}\} = [R]^T \{u\} \]

where \([R]\) is the transformation between the global coordinates \( XYZ \) and the local coordinates \( xyz \). The elements in \([R]\) are:

\[ [R] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \]

(2.7)

For the case of zero coning and zero pitch, \([R]\) is equal to \([T_{EP}]\), as in Eq. 2.3. Also, the
unit vector \( \{ s \} \) is:
\[
\{ s \} = [\alpha_{31}, \alpha_{32}, \alpha_{33}]^T
\]  
(2.8)

The position vector of a point on the deformed beam with reference to frame \( XYZ \) is:
\[
\{ r \} = \{ r_0 \} + \{ \bar{u} \} = \{ r_0 \} + [R]^T \{ u \}
\]  
(2.9)

and:
\[
\{ \dot{r} \} = [R]^T \{ \dot{u} \}, \text{ because } \{ \dot{r}_0 \} = 0
\]  
(2.10)

### 2.2.3 Lagrange’s Equation

Lagrange’s equation in vector form is:
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \{ \dot{u} \}} \right) - \frac{\partial T}{\partial \{ u \}} + \frac{\partial U}{\partial \{ u \}} = \{ F \}
\]  
(2.11)

where \( T \) and \( U \) are the kinetic and strain energies, respectively, and \( \{ F \} \) is the generalized force vector. For the free vibration problem, \( \{ F \} = 0 \). The kinetic energy of the beam is:
\[
T = \frac{1}{2} \int \rho A \{ v \}^T \{ v \} ds
\]  
(2.12)

where \( \rho \) is the mass density and \( A \) is the sectional area. The strain energy of the beam is:
\[
U = \frac{1}{2} \int EA \left( \frac{\partial w}{\partial s} \right)^2 ds + \frac{1}{2} \int EI_x \left( \frac{\partial^2 v}{\partial s^2} \right)^2 ds + \frac{1}{2} \int EI_y \left( \frac{\partial^2 u}{\partial s^2} \right)^2 ds + \frac{1}{2} \int F(s) \left( \frac{\partial u}{\partial s} \right)^2 ds + \frac{1}{2} \int F(s) \left( \frac{\partial v}{\partial s} \right)^2 ds
\]  
(2.13)

At this point the analysis does not include torsion of the element, which is assumed in Leung and Fung [1] to be uncoupled from rotation. The first three terms in the strain energy
equation 2.14 are the standard beam extension and bending terms, which are found in finite element texts such as Rao’s [28]. The last two terms account for the work done by the element axial force $F(s)$, which arises from rotation. They account for the beam’s resistance to bending due to centrifugal forces, and are sometimes called axial reduction factors. Their derivation, described in Section 3.3.1, are from the theory of vibrating strings, which Meirovitch [41] describes in the context of rotating beams.

Substituting Eqs. 2.9 and 2.10 into Eq. 2.6 the absolute velocity vector is:

$$\{v\} = [R]^T\{\dot{u}\} + [\Omega]\{r_0\} + [R]^T\{u\}$$ (2.14)

The term in the kinetic energy equation 2.12 is therefore:

$$\{v\}^T\{v\} = \{\dot{u}\}^T[R][R]^T\{\dot{u}\} + \{u\}^T[R][\Omega]^T[\Omega][R]^T\{u\} + \{r_0\}^T[\Omega]^T[\Omega]\{r_0\} + 2\{\dot{u}\}^T[R][\Omega]\{r_0\} + 2\{u\}^T[R][\Omega]^T[\Omega]\{u\}$$ (2.15)

From Eqs. 2.7 and 2.5:

$$[R][R]^T = [I] \quad \text{and} \quad [\Omega]^T[\Omega] = [\Omega^2] = \Omega^2$$

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix}$$ (2.16)

For the finite element method, the element displacement vector $\{u_e\}$ is interpolated from the nodal coordinate vector $\{q_e\}$ as in:

$$\{u_e\} = [N]\{q_e\} \quad \text{and} \quad \{\dot{u}_e\} = [N]\{\dot{q}_e\}$$ (2.17)
where $N$ is the shape function matrix:

$$
[N] = \begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix}
$$

(2.18)

Section 2.2.6 describes the development of these $N$ terms. The nodal coordinate vector $\{q_e\}$ is:

$$
\{q_e\} = [\{q_1\}^T, \{q_2\}^T, \{q_3\}^T]^T
$$

(2.19)

where:

$$
\{q_1\}^T = [u_1, \theta_{y1}, u_2, \theta_{y2}]
$$

(2.20)

are the bending nodal displacements in the $xz$ plane, and:

$$
\{q_2\}^T = [v_1, \theta_{x1}, v_2, \theta_{x2}]
$$

(2.21)

are the bending nodal displacements in the $yz$ plane, and:

$$
\{q_3\}^T = [w_1, w_2]
$$

(2.22)

are the axial nodal displacements.

From Eqs. 2.12 and 2.16 the kinetic energy for an element is:

$$
T_e = \frac{1}{2} \{q_e\}^T [M_e] \{q_e\} + \frac{1}{2} \{q_e\}^T [K_{\Omega_e}] \{q_e\} + T_{0e} +
\{\dot{q}_e\}^T \{f_e\} + \{\dot{q}_e\}^T \{G_e\} \{q_e\} + \{F_{\Omega_e}\}^T \{q_e\}
$$

(2.23)
where (following Leung and Fung [1]):

\[
\begin{align*}
[M_e] &= \int_0^l \rho A[m] ds, \\
[K_{\Omega e}] &= \int_0^l \rho A[k_{\Omega}] ds, \\
[G_e] &= \int_0^l \rho A[g] ds, \\
\{f_e\} &= \int_0^l \rho A[N]^T[R]\{\Omega\}\{r_0\} ds, \\
\{F_{\Omega e}\} &= \int_0^l \rho A[N]^T[f_{\Omega e}, f_{\Omega e}, f_{\Omega e}]^T ds, \\
T_{0e} &= \frac{1}{2} \int_0^l \rho A\{r_0\}^T[\Omega^2]\{r_0\} ds
\end{align*}
\] (2.24)

where \(M_e\) is the element mass matrix, \(K_{\Omega e}\) is the element spin-stiffness matrix, and \(G_e\) is the element gyroscopic matrix. The vectors \(\{f_e\}\) and \(\{F_{\Omega e}\}\) and the scalar \(T_{0e}\) drop out of the ensuing analysis. The terms in the element matrices are:

\[
\begin{align*}
[m] &= \begin{bmatrix}
N_1^T N_1 \\
N_2^T N_2 \\
N_3^T N_3
\end{bmatrix} \\
[g] &= \Omega \begin{bmatrix}
0 & b_1 N_1^T N_2 & b_2 N_1^T N_3 \\
b_1^T N_2 & 0 & b_3 N_2^T N_3 \\
skew\, symmetric & 0 & 0
\end{bmatrix} \\
[k_{\Omega}] &= \Omega^2 \begin{bmatrix}
a_{11} N_1^T N_1 & a_{12} N_1^T N_2 & a_{13} N_1^T N_3 \\
a_{12}^T N_2 & a_{22} N_2^T N_2 & a_{23} N_2^T N_3 \\
symmetric & a_{33} N_3^T N_3
\end{bmatrix}
\end{align*}
\] (2.25)

where:

\[
\begin{align*}
a_{11} &= \alpha_{12}^2 + \alpha_{13}^2, & a_{12} &= \alpha_{12} \alpha_{22} + \alpha_{13} \alpha_{23}, \\
a_{13} &= \alpha_{12} \alpha_{32} + \alpha_{13} \alpha_{33}, & a_{22} &= \alpha_{22}^2 + \alpha_{23}^2, \\
a_{23} &= \alpha_{22} \alpha_{32} + \alpha_{23} \alpha_{33}, & a_{22} &= \alpha_{22}^2 + \alpha_{23}^2
\end{align*}
\] (2.26)
and:

\[
\begin{align*}
    b_1 &= \alpha_{13}\alpha_{22} - \alpha_{12}\alpha_{23}, \\
    b_2 &= \alpha_{13}\alpha_{32} - \alpha_{12}\alpha_{33}, \\
    b_3 &= \alpha_{23}\alpha_{32} - \alpha_{22}\alpha_{33}
\end{align*}
\] (2.27)

The total kinetic energy comes from assembly of the finite element matrices in typical fashion, as described in Rao [28], with the matrices transformed to the \(XYZ\) coordinate systems and the common nodes added for the compatibility relation. The total kinetic expression is:

\[
T = \sum_e T_e = \frac{1}{2} \{\dot{q}\}^T [M]\{\dot{q}\} + \frac{1}{2} \{\dot{q}\}^T [K_\Omega]\{\dot{q}\} + T_0 + \{\dot{q}\}^T \{f\} + \{\dot{q}\}^T [G]\{q\} + \{F_\Omega\}^T \{q\},
\] (2.28)

where \(\{q\}\) is the global nodal displacement vector in the rotating reference frame \(XYZ\).

Similar to the kinetic energy, the total strain energy for an element is:

\[
U_e = \frac{1}{2} \{q_e\}^T ([K_{ee}] + [K_{ge}]) \{q_e\},
\] (2.29)

where:

\[
[K_{ee}] = \begin{bmatrix}
    EI_x[K_a] \\
    EI_y[K_b] \\
    EA[K_c]
\end{bmatrix},
\]

\[
[K_{ge}] = \begin{bmatrix}
    [K_{ga}] \\
    [K_{gb}]
\end{bmatrix},
\] (2.30)
where $K_{ee}$ is the element elastic stiffness matrix and $K_{ge}$ is the element axial force stiffness matrix. The terms in the element matrices are:

$$
[K_a] = \int_0^l [N_{11}'']^T [N_{11}''] ds,
[K_b] = \int_0^l [N_{22}'']^T [N_{22}''] ds,
[K_c] = \int_0^l [N_{33}']^T [N_{33}'] ds,
$$

$$
[K_{ga}] = \int_0^l F(s) [N_{11}']^T [N_{11}'] ds,
[K_{gb}] = \int_0^l F(s) [N_{22}']^T [N_{22}'] ds,
$$

where the prime denotes differentiation with respect to $s$.

The total strain energy is:

$$
U = \sum_e U_e = \frac{1}{2} \{ q \}^T ([K_e] + [K_g]) \{ q \}.
$$

### 2.2.4 Steady State Equations and Axial Force

The nodal displacement vector $\{ q \}$ is divided into the steady state displacement $\{ q_s \}$ and the oscillation about steady state, $\{ q_v \}$, so that:

$$
\{ q \} = \{ q_s \} + \{ q_v \}
$$

For steady state the oscillatory component and the derivatives are zero, as in:

$$
\{ q_v \}, \{ q_v \}, \{ q_s \}, \text{ and } \{ \dot{q} \} = 0
$$

Inserting Eqs. 2.33 and 2.34 into Eqs. 2.28 and 2.32 the steady state kinetic energy becomes:

$$
T = \frac{1}{2} \{ q_s \}^T [K_{\Omega}] \{ q_s \} + T_0 + \{ F_{\Omega} \}^T \{ q_s \},
$$
and the steady state strain energy becomes:

\[ U = \frac{1}{2} \{q_s\}^T ([K_e] + [K_g]) \{q_s\}. \]  \hspace{1cm} (2.36)

The Lagrangian equation 2.11 then reduces to:

\[- \frac{\partial T}{\partial \{q_s\}} + \frac{\partial U}{\partial \{q_s\}} = 0, \]  \hspace{1cm} (2.37)

and with Eqs. 2.35 and 2.36:

\[ ([K_e] + [K_g] - [K_\Omega]) \{q_s\} = \{F_\Omega\}. \]  \hspace{1cm} (2.38)

The blade is assumed to be a cantilever beam and therefore the internal forces are predetermined. The axial force stiffness matrix $[K_g]$ is therefore independent of the steady state deformation. The derivation of the axial force $F(s)$ assumes that the centrifugal force does not change with deflection. From Leung and Fung [1] the centrifugal force per unit length along the beam is:

\[ \tau = -\rho A \{s\}^T [\Omega][\Omega] \{r_0\}. \]  \hspace{1cm} (2.39)

From Eqs. 2.4, 2.5, and 2.8, the above equation becomes:

\[ \tau = \rho A (a + bs) \]  \hspace{1cm} (2.40)

where:

\[ a = \Omega^2 (\alpha_{32} Y_0 + \alpha_{33} Z_0) \]  \hspace{1cm} (2.41)

and:

\[ b = \Omega^2 (\alpha_{32}^2 + \alpha_{33}^2) \]  \hspace{1cm} (2.42)

and $Y_0$ and $Z_0$ are the coordinates of the element inboard endpoint $\{r_g\}$. The axial force at
a point along the element, \( F(s) \) (Figure 2.4) is:

\[
F(s) = F_0 - \int_0^s \tau(s) \, ds
\]  

(2.43)

where \( F_0 \) is the axial force at the inboard node of the element.

![Figure 2.4: Beam element axial force (after Leung [1])](image)

Substituting Eq. 2.40 into Eq. 2.43:

\[
F(s) = F_0 - \int_0^s \rho A(a + bs) \, ds
= F_0 - \int_0^s \mu(s)(a + bs) \, ds
\]  

(2.44)

where \( \mu(s) \) is the lineal density (kg/m) along the element. The model assumes linearly tapered properties, so the lineal density is:

\[
\mu(s) = \left[ \mu_0 + \frac{\mu_l - \mu_0}{l} s \right]
\]  

(2.45)

where the zero subscript represents the property at the inboard node, and the \( l \) subscript represents the property at the outboard node. Substituting Eq. 2.45 into Eq. 2.44 and integrating:

\[
F(s) = F_0 - cs - \frac{ds^2}{2} - \frac{es^3}{3}
\]  

(2.46)
where:

\[ c = \mu_0 a \]
\[ d = \frac{\mu_l - \mu_0}{l} a + \mu_0 bl \]
\[ e = \frac{\mu_l - \mu_0}{l} b \]  \hspace{1cm} (2.47)

### 2.2.5 Oscillation about Steady State

For a small oscillation about steady state, the nodal velocity is:

\[ \{ \dot{q} \} = \{ q_v \}, \]  \hspace{1cm} (2.48)

and the kinetic and strain energies are:

\[
T = \frac{1}{2} \{ q_v \}^T [M] \{ q_v \} + \frac{1}{2} \{ q_s + q_v \}^T [K_g] \{ q_s + q_v \} + T_0 + \{ q_v \}^T \{ f \} + \{ q_v \}^T [G] \{ q_s + q_v \} \{ F_d \}^T \{ q_s + q_v \},
\]  \hspace{1cm} (2.49)

and:

\[
U = \frac{1}{2} \{ q_s + q_v \}^T \{ \{ K_s \} + [K_g] \} \{ q_s + q_v \}.
\]  \hspace{1cm} (2.50)

Substituting these into Eq. 2.11, the Lagrangian becomes:

\[
[M] \{ \ddot{q}_v \} + 2[G] \{ \dot{q}_v \} + (\{ K_c \} + [K_g] - [K_d]) \{ q_v \} = \{ 0 \}.
\]  \hspace{1cm} (2.51)
2.2.6 Beam Element Matrices

The beam element matrices are assembled using Eqs. 2.24 and 2.30. The inertia and stiffness properties vary linearly along the element as in Eq. 2.45 for the mass and:

\[
EA(s) = \left[ EA_0 + \frac{EA_l - EA_0}{l} s \right],
\]

(2.52)

for the extensional stiffness, and:

\[
EI(s) = \left[ EI_0 + \frac{EI_l - EI_0}{l} s \right],
\]

(2.53)

for the bending stiffnesses, and:

\[
GJ(s) = \left[ GJ_0 + \frac{GJ_l - GJ_0}{l} s \right],
\]

(2.54)

for the torsional stiffness, and:

\[
J'(s) = \left[ J'_0 + \frac{J'_l - J'_0}{l} s \right],
\]

(2.55)

for the polar moment of inertia about the centroidal axis.

The shape function matrices are:

\[
[N_1] = \begin{bmatrix}
(2s^3 - 3ls^2 + l^3)/l^3, (s^3 - 2ls^2 + l^2s)/l^2, \\
-(2s^3 - 3ls^2)/l^3, (s^3 - ls^2)/l^2
\end{bmatrix}
\]

(2.56)

and:

\[
[N_2] = \begin{bmatrix}
(2s^3 - 3ls^2 + l^3)/l^3, -(s^3 - 2ls^2 + l^2s)/l^2, \\
-(2s^3 - 3ls^2)/l^3, -(s^3 - ls^2)/l^2
\end{bmatrix}
\]

(2.57)
and:

\[ [N_3] = [(1 - s/l), s/l] \] (2.58)

where \( l \) as the length of the element. These are standard shape (or interpolation) functions for a space frame element, which can be found in Rao’s finite element text [28]. The nodal degrees of freedom are ordered as in Figure 2.5

![Figure 2.5: Nodal degrees of freedom](image)

Appendix A list the entries for the beam element matrices.

### 2.2.7 Solution for Blade Frequencies and Mode Shapes

The model frequencies and mode shapes are from an eigenvalue solution of Equation 2.51. This equation cannot be solved with standard techniques because of the skew-symmetric gyroscopic matrix. Meirovitch [42] outlines a solution that avoids complex arithmetic, which combines the matrices into symmetric ones. Given the stiffness matrix:

\[ K = [K_e] + [K_g] - [K_\Omega], \] (2.59)
Meirovitch makes a modified stiffness matrix:

\[
K^* = \begin{bmatrix}
KM^{-1}K & KM^{-1}K \\
G^TM^{-1}K & K + G^TM^{-1}G
\end{bmatrix},
\]

(2.60)

and a modified mass matrix:

\[
M^* = \begin{bmatrix}
K & 0 \\
0 & M
\end{bmatrix},
\]

(2.61)

which assembles into a standard eigenvalue problem with \(2n\) entries:

\[
K^* \{x_r\} = \lambda_r M^* \{x_r\},
\]

(2.62)

where \(n\) is the number of degrees of freedom, \(\{x_r\}\) is an eigenvector \((r = 1, 2, \ldots, 2n)\), and \(\lambda_r\) is an eigenvalue. The solutions consist of \(n\) pairs of repeated eigenvalues and \(n\) pairs of associated eigenvectors \(y_r\) and \(z_r\), where \(y_r\) is the real part and \(z_r\) is the imaginary part of eigenvector \(x_r\).

The author used Meirovitch’s method for verification with Leung’s results, which is described in the Section 2.3.4. However, to use the modes from Meirovitch’s method for solving the response, a dynamics code would require twice the number of degrees of freedom; one mode for the real part and one mode for the complex part. Therefore the final method in CurveFEM assumed that the gyroscopic terms for the wind turbine blade are considered small. This is standard practice in the industry. The blade motions perpendicular to the axis of rotation are small in comparison to motions parallel to the axis, as in blade flap motion. Therefore, the analysis neglects the gyroscopic matrix and the eigensolution for Equation 2.51 is:

\[
K \{x_r\} = \lambda_r M \{x_r\},
\]

(2.63)

with \(K\) from Eq. 2.59. The solutions consist of \(n\) eigenvalues and \(n\) real eigenvectors. The author verified this method for a rotating, tapered beam with published results from Baner-
jee, Su, and Jackson [43], which Section 2.3.3 describes. Other verifications described below are the deflection and vibration of a non-rotating curved beam, and the vibration of a rotating curved beam. Appendix B describes the finite element program structure.

2.3 Finite Element Method Verification Results

2.3.1 Curved Beam Deflection Verification

In following Kosmatka’s work [2], the author verified the finite element stiffness matrices (K_e) with an example curved beam under deflection. Figure 2.6 shows the cantilever beam. The beam had six elements with applied forces at the end in the x and y directions.

Material: Aluminum 6061-T6

The finite element results compare well with analytical results from Roark’s [44] for beams
of varying curvature, shown in Figure 2.7 for the $x$-force direction and Fig. 2.8 for the $y$-force direction.

Figure 2.7: Curve beam deflection and slope for $F_x = 100$ lb
Figure 2.8: Curve beam deflection and slope for $F_y = 100$ lb
2.3.2 Non-Rotating Curved Beam Modes Verification

Again following Kosmatka’s work [2], the author verified the finite element stiffness and mass matrices (\(K_e\) and \(M\)) with an example of curved beam free-response. Figure 2.9 shows the curved beam with fixed-fixed boundary conditions. Table 2.2 shows the finite element results compared to Kosmatka [2] and Blevins [45]. The results are in good agreement with previous results for curved beams.

![Curved beam diagram](image)

Table 2.2: Curved beam modes verification

<table>
<thead>
<tr>
<th>Description</th>
<th>Frequency from Blevins [45] (Hz)</th>
<th>Frequency from Kosmatka [2] (Hz)</th>
<th>Finite element</th>
</tr>
</thead>
<tbody>
<tr>
<td>First out-of-plane</td>
<td>473.2</td>
<td>472.2</td>
<td>475</td>
</tr>
<tr>
<td>Second out-of-plane</td>
<td>-</td>
<td>1343.8</td>
<td>1355</td>
</tr>
<tr>
<td>First in-plane</td>
<td>2277.5</td>
<td>2237.3</td>
<td>2231</td>
</tr>
</tbody>
</table>

2.3.3 Rotating Tapered Beam Verification

The author then conducted simulations with a tapered rotating beam to compare to work by Banerjee, Su, and Jackson [43]. Banerjee et al uses the dynamic stiffness method
for solution, neglects the Coriolis terms and only considers the flapping (or out-of-plane) motion. The author’s verification also neglected the Coriolis matrix, and did not use Meirovitch’s method for solution. Instead, the model used the standard eigenvalue solution as in Eq. 2.63.

The verification used the beam with dimensions and properties shown in Figure 2.10.

Figure 2.10: Rotating tapered beam (all dimensions in inches)

The author also compared with results for a non-spinning tapered beam by Mabie and Rogers [46] and with an equivalent model with the NREL MODES program [29], described in Section 2.1. The verification was conducted at rotational speeds representing the range of Banerjee’s et al analysis. Figure 2.11 shows the verification results for the first three modes, which show good agreement amongst the methods.
Figure 2.11: Tapered beam verification results
2.3.4 Rotating Beams from Leung and Fung 1988 Verification

The author compared the finite element program against results from Leung [1] for several examples. The first example was for a rotating horizontal cantilever (Figure 2.12) of four elements.

![Figure 2.12: Horizontal Cantilever (after Leung [1])]()

The verification runs were at zero rotational speed and a non-dimensional spinning rate of 20 shown in Figure 2.13, with the lines digitized from Leung [1] and the triangles representing the author’s results.

The results show good agreement at zero rotational speed, which is comparable to results available in Roark’s [44]. For the maximum rotational speed, the first two bending modes match well, but for CurveFEM the third and fourth modes have slightly lower natural frequency.
Figure 2.13: Horizontal Cantilever Verification Results
The next example is a rotating beam inclined $45^\circ$ from the rotational axis, shown in Figure 2.14.

![Figure 2.14: Inclined Cantilever (after Leung [1])](image)

Again the results show good agreement at zero rotational speed. For the maximum rotational speed, for CurveFEM the second and fourth modes show lower frequencies (Figure 2.15).
Figure 2.15: Inclined Cantilever Verification Results
The next two examples are for an “L”-beam shown in Figure 2.16

![Diagram of an L-beam](Figure 2.16: L-beam (after Leung [1]))

The results show good agreement at zero rotational speed. For the 90° beam (Figure 2.17), the second and third modes at the highest rotational speed show lower natural frequency for CurveFEM.

For the 30° beam (Figure 2.18), the results show good agreement for the first and second modes at the highest rotational speed, and slightly lower frequency for the third mode for CurveFEM.
Figure 2.17: L-beam with $\theta = 90^\circ$ Verification Results
Figure 2.18: L-beam with $\theta = 30^\circ$ Verification Results
2.3.5 Rotating Curved Beam Verification

The author verified CurveFEM with rotating curved beam results from Wang and Mahrenholtz [47]. This is the only reference the author found for modes of rotating curved beams. Wang and Mahrenholtz developed a model for bending of the minor axis of a rotating curved beam with the Bernoulli-Euler approximation and with Coriolis terms neglected. The authors used a Galerkin method for solution. The author compared results for straight beams and beams with inplane curvature of 0.6 (arclength/radius) and zero hub radius. The beam properties were according to the following relation:

\[
\frac{\mu}{EI} L^4 = 1,
\]

where \( L \) is the total beam length. The author used 20 elements of equal length for the CurveFEM model. The major axis bending, torsional, and extensional stiffnesses were 1000 times the value of the inplane bending stiffness. Figure 2.19 shows the comparison. As in Wang and Mahrenholtz [47], the CurveFEM results show little difference in frequency between straight and curved beams. The results match well for the first mode. For the second mode, both analysis predict lower frequencies for the curved blade; however, CurveFEM predicts lower frequencies overall.
Figure 2.19: Rotating curved beam verification results
2.4 Conclusions and Recommendations

The verification results show excellent agreement between the author’s finite element analysis and previously published results for curved beam deflection and natural frequencies. The results also show excellent agreement with previous work on rotating tapered beams. For rotating beams that are not straight, the results show good agreement for the fundamental mode; however, CurveFEM underpredicts the second bending mode frequency. This may be due to differences in the solution method. Wang and Mahrenholtz [47] do not mention their numerical procedure, and Leung and Fung [1] use a Newtonian procedure. Without experimental results, it is not clear which method is most correct.

For future work, the author recommends the following:

- Add pitch angle to the analysis
- Add coning angle to the analysis
- Add center of mass and elastic axis offsets to the analysis
- Verify natural frequencies and mode shapes with experimental data on rotating curved beams

The following chapter describes how the author used the CurveFEM-derived mode shapes and frequencies in a modified wind-turbine dynamics code.
Chapter 3

Dynamic Analysis of Curved Wind Turbine Blades

3.1 Introduction

This chapter presents the development of the equations of motion for the curved blade to be used in the wind turbine analysis code FAST. The new program is called CurveFAST to distinguish it from the existing version of FAST. This development is separate from the finite element modeling of the blade that is used to solve for the blade mode shapes, as explained in Chapter 2. These mode shapes enter in the equations of motion, shown below.

This development follows the theoretical development of FAST. Currently there is no FAST theory manual, however, the author obtained from the developer, Jason Jonkman of NREL, several documents outlining the equations of motion for FAST. Much of the background is also documented in Jonkman’s master’s thesis [26], which worked from the original code developed at Oregon State University [23]. The basis of the method is known as “Kane’s Method,” which can be found in Kane and Levinson’s text [25].

FAST uses the aerodynamic subroutines of AeroDyn [17] for calculation of the aerodynamic forces. Morariety and Hansen [48] cover the theory for AeroDyn. AeroDyn uses
blade element momentum (BEM) theory but has several improvements to account for unsteadiness and asymmetric loading in wake. For CurveFAST, the author modified version 6.10a of FAST that used version 12.58 of AeroDyn.

The following development begins with defining coordinate systems, then moves systematically to kinematics and then kinetics.

### 3.2 Coordinate Systems and Transformations

The section describes the coordinate systems unique to CurveFAST. They are similar to those from Section 2.2.1, however, the transformations are different in order to match those used by the FAST2ADAMS.f90 pre-processor. The author retained the FAST2ADAMS.f90 transformations because CurveFAST would be verified with Adams.

#### 3.2.1 Local Undeflected Axis

The first coordinate system unique to CurveFAST is the blade element system aligned with the local undeflected-axis with unit vectors $L_{ij}^{B1} (i = 1, 2, 3)$. The superscript $B1$ refers to blade number one, and there are similar coordinate systems for the other blades. This coordinate system is transformed from the pitched coordinate system, with unit vectors $j_1^{B1}, j_2^{B1},$ and $j_3^{B1}$. The transformation is:

$$
\begin{bmatrix}
    L_{j_1}^{B1}(r) \\
    L_{j_2}^{B1}(r) \\
    L_{j_3}^{B1}(r)
\end{bmatrix} = [T_{EP}(r)]
\begin{bmatrix}
    j_1^{B1} \\
    j_2^{B1} \\
    j_3^{B1}
\end{bmatrix} \quad (3.1)
$$

The first stage involves a transformation matrix $[\lambda_1]$ between the pitched coordinates $XYZ$ and the coordinates $\bar{x} \, \bar{y} \, \bar{z}$ (Fig. 3.1):
The next step is between the local coordinates $xyz$ (principal axes) and the coordinates $\bar{x}\bar{y}\bar{z}$ as (Fig. 3.2):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\lambda_2] \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}$$

The desired transformation between the $xyz$ and the $XYZ$ system is therefore:

$$[T_{EP}(r)] = [\lambda_2][\lambda_1]$$
Figure 3.2: Second transformation of element with built-in structural twist

where:

\[
\begin{pmatrix}
  x \\
  y \\
  z \\
\end{pmatrix} = [T_{EP}(r)] \begin{pmatrix}
  X \\
  Y \\
  Z \\
\end{pmatrix}
\]

The first transformation, \( \lambda_1 \) assumes small rotations about the \( X \) and \( Y \) axis, with:

\[
\lambda_1 = [TransMat(\theta_1 = \theta_{BI}^X(r), \ \theta_2 = \theta_{BI}^Y(r), \ 0)]
\]

where \( TransMat \) is the orthonormal transformation matrix subroutine used in FAST for small rotations about the three axes. The rotations at a particular analysis node \( i \) are:

\[
\begin{align*}
\theta_{BI}^X_i &= \frac{X_{i+1} - X_{i-1}}{Z_{i+1} - Z_{i-1}} \\
\theta_{BI}^Y_i &= \frac{Y_{i+1} - Y_{i-1}}{Z_{i+1} - Z_{i-1}}
\end{align*}
\]

where \( X \) is the blade precurve (RefAxisxb), \( Y \) is the blade presweep (RefAxisyb) and \( Z \) is the length along the \( j_3 \) axis RNodes.
After the first transformation, the principal axes of the element (xyz axes) make an angle $\theta_s$ (built-in structural twist) about the negative $\bar{z}$ axis. This direction is such that a positive $\theta_s$ results in a lower angle of attack. The transformation between the systems is:

$$
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \begin{bmatrix}
  \cos \theta_s & -\sin \theta_s & 0 \\
  \sin \theta_s & \cos \theta_s & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  \bar{x} \\
  \bar{y} \\
  \bar{z}
\end{bmatrix} = [\lambda_2] \begin{bmatrix}
  \bar{x} \\
  \bar{y} \\
  \bar{z}
\end{bmatrix}
$$

so that:

$$
[\lambda_2] = \begin{bmatrix}
  \cos \theta_s & -\sin \theta_s & 0 \\
  \sin \theta_s & \cos \theta_s & 0 \\
  0 & 0 & 1
\end{bmatrix}
$$

### 3.2.2 Local Deflected Coordinate System

The next transformation is from the undeflected element to the deflected coordinate system, which is:

$$
\begin{bmatrix}
  n^B_1(r) \\
  n^B_2(r) \\
  n^B_3(r)
\end{bmatrix} = [T_{DE}(r)] \begin{bmatrix}
  L^B_1(r) \\
  L^B_2(r) \\
  L^B_3(r)
\end{bmatrix}
$$

with

$$
[T_{DE}(r)] = [TransMat(\theta_1 = \theta_{Lx}^B(r), \theta_2 = \theta_{Ly}^B(r), \theta_3 = \theta_{Lz}^B(r))]
$$

The rotations, which are angular deflections, are:

$$
\begin{bmatrix}
  \theta^B_{Lx}(r) \\
  \theta^B_{Ly}(r) \\
  \theta^B_{Lz}(r)
\end{bmatrix} = [T_{EP}(r)] \begin{bmatrix}
  \theta^B_{jx}(r) \\
  \theta^B_{jy}(r) \\
  \theta^B_{jz}(r)
\end{bmatrix}
$$

(3.2)
The rotations in the \( j \) system (\( \theta_{jx}, \theta_{jy}, \theta_{jz} \)) are:

\[
\begin{align*}
\theta_{jx}^{BI}(r) &= \Psi_{4,1}^{BI}(r)\Phi_4(r) * q_{B1F1} + \Psi_{4,2}^{BI}(r)\Phi_4(r) * q_{B1F2} \\
&\quad + \Psi_{4,3}^{BI}(r)\Phi_4(r) * q_{B1E1} + \Psi_{4,4}^{BI}(r)\Phi_4(r) * q_{B1T1} \\
\theta_{jy}^{BI}(r) &= \Psi_{5,1}^{BI}(r)\Phi_5(r) * q_{B1F1} + \Psi_{5,2}^{BI}(r)\Phi_5(r) * q_{B1F2} \\
&\quad + \Psi_{5,3}^{BI}(r)\Phi_5(r) * q_{B1E1} + \Psi_{5,4}^{BI}(r)\Phi_5(r) * q_{B1T1} \\
\theta_{jz}^{BI}(r) &= \Psi_{6,1}^{BI}(r)\Phi_6(r) * q_{B1F1} + \Psi_{6,2}^{BI}(r)\Phi_6(r) * q_{B1F2} \\
&\quad + \Psi_{6,3}^{BI}(r)\Phi_6(r) * q_{B1E1} + \Psi_{6,4}^{BI}(r)\Phi_6(r) * q_{B1T1}
\end{align*}
\]

(3.3)

In these rotation equations, as an example, the symbol \( \Psi_{4,1}^{BI}(r) \) is the eigenvector component (subscript-prefix 4) in the pitched \( j \) system for the x-component of slope (“sweep” deflection) for the first blade flap mode (subscript-suffix 1) at a particular radial station for blade 1. The symbol \( \Phi_4(r) \) represents the interpolation function for the x-component of slope. The CurveFAST “analysis” nodes coincide with the finite element nodes and the interpolation functions are equal to 1. In the current version of the code the variable \( \Phi \) was not be implemented. The symbol \( q_{B1E1} \) is the first flap mode degree of freedom for blade 1. For reference, Fig. 2.5 shows the numbering of the element degrees of freedom. Table 3.1 shows CurveFAST’s degrees of freedom for blade 1.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_{B1F1} )</td>
<td>Blade 1, First Flap Mode</td>
</tr>
<tr>
<td>( q_{B1F2} )</td>
<td>Blade 1, Second Flap Mode</td>
</tr>
<tr>
<td>( q_{B1E1} )</td>
<td>Blade 1, First Edge Mode</td>
</tr>
<tr>
<td>( q_{B1T1} )</td>
<td>Blade 1, First Torsion Mode</td>
</tr>
</tbody>
</table>

To reduce the number of multiplications during the simulation, the program uses Eq.
3.2 in the form:

\[
\begin{bmatrix}
\theta^{B1}_{Lx}(r) \\
\theta^{B1}_{Ly}(r) \\
\theta^{B1}_{Lz}(r)
\end{bmatrix}
= \begin{bmatrix} T_{EP}(r) \end{bmatrix} \begin{bmatrix}
\Psi
\end{bmatrix}
\begin{bmatrix}
q_{B1F1} \\
q_{B1F2} \\
q_{B1E1} \\
q_{B1T1}
\end{bmatrix}
\]

with the matrix \([\Psi]\) as:

\[
[\Psi] = \begin{bmatrix}
\Psi^{B1}_{4,1}(r) & \Psi^{B1}_{4,2}(r) & \Psi^{B1}_{4,3}(r) & \Psi^{B1}_{4,4}(r) \\
\Psi^{B1}_{5,1}(r) & \Psi^{B1}_{5,2}(r) & \Psi^{B1}_{5,3}(r) & \Psi^{B1}_{5,4}(r) \\
\Psi^{B1}_{6,1}(r) & \Psi^{B1}_{6,2}(r) & \Psi^{B1}_{6,3}(r) & \Psi^{B1}_{6,4}(r)
\end{bmatrix}
\]

The subroutine `CalcT_EP` in the file `FAST_IO_SML.f90` pre-multiplies the matrices \([T_{EP}(r)]\) and \([\Psi]\) at initialization.

### 3.2.3 Local Aerodynamics Coordinate System

The coordinate system for calculating and returning the aerodynamic loads is:

\[
\begin{bmatrix}
\mathbf{m}^{B1}_{1}(r) \\
\mathbf{m}^{B1}_{2}(r) \\
\mathbf{m}^{B1}_{3}(r)
\end{bmatrix}
= \begin{bmatrix}
\cos \left[ \beta_p^{B1} + \theta_s^{B1}(r) \right] & \sin \left[ \beta_p^{B1} + \theta_s^{B1}(r) \right] & 0 \\
-\sin \left[ \beta_p^{B1} + \theta_s^{B1}(r) \right] & \cos \left[ \beta_p^{B1} + \theta_s^{B1}(r) \right] & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{n}^{B1}_{1}(r) \\
\mathbf{n}^{B1}_{2}(r) \\
\mathbf{n}^{B1}_{3}(r)
\end{bmatrix}
\]
This is the same transformation as in FAST, with $\beta_p$ representing the blade pitch and $\theta_s$ representing the built-in structural twist (both positive to lower angle of attack). The transformation does not include the $\theta_{Lz}$ (elastic twist) term. The transformation to the trailing-edge coordinate system, used in aeroacoustic calculations, is also the same as FAST, with:

$$
\begin{align*}
\begin{bmatrix}
\mathbf{te}_{1}^{B1}(r) \\
\mathbf{te}_{2}^{B1}(r) \\
\mathbf{te}_{3}^{B1}(r)
\end{bmatrix} &=
\begin{bmatrix}
\cos \left[ \beta_{p}^{B1} + \theta_{a}^{B1}(r) \right] & -\sin \left[ \beta_{p}^{B1} + \theta_{a}^{B1}(r) \right] & 0 \\
\sin \left[ \beta_{p}^{B1} + \theta_{a}^{B1}(r) \right] & \cos \left[ \beta_{p}^{B1} + \theta_{a}^{B1}(r) \right] & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{m}_{1}^{B1}(r) \\
\mathbf{m}_{2}^{B1}(r) \\
\mathbf{m}_{3}^{B1}(r)
\end{bmatrix}
\end{align*}
$$

where $\theta_a$ is the local aerodynamic twist, a user input, which is usually equal to the structural twist.

### 3.3 Kinematics

There are several references to the model degrees of freedom in CurveFAST in this section; for example, the indices in the partial velocity derivatives refer to the degrees of freedom. For reference, Table 3.2 shows these degrees of freedom for 2- and 3-bladed wind turbines.
Table 3.2: CurveFAST Degrees of Freedom

<table>
<thead>
<tr>
<th>No.</th>
<th>2 Bladed</th>
<th>3 Bladed</th>
<th>Description</th>
</tr>
</thead>
<tbody>
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<td>B3T1</td>
<td>Blade 3 First Torsion Mode</td>
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3.3.1 Positions and Displacements

The position vector from the apex of the coned system (point \(Q\)) to the blade 1 node (\(SI\)) is:

\[
\mathbf{r}^{QS1}(r) = \begin{cases} 
\{ h^B_1(r), h^B_2(r), \text{hub radius} + h^B_3(r) \} \\
\frac{j^B_1}{j^B_2} \\
\frac{j^B_3}{j^B_3} \end{cases}
\]

\[
+ \begin{cases} 
\{ u^B(r), v^B(r), w^B(r) + w^B_{AxRed}(r) \} \\
\frac{j^B_1}{j^B_2} \\
\frac{j^B_3}{j^B_3} \end{cases}
\]

(3.4)

where \(h_1, h_2,\) and \(h_3\) are the undeflected blade element offsets shown in Fig. 3.3. In the code, the offsets are given by \(\text{RefAxis}x_b(i), \text{RefAxis}y_b(i),\) and \(\text{RNodesNorm}(i) \times \text{BldFlexL}\) respectively. Also in the equation, \(u, v,\) and \(w\) are the deflections in the \(j\) system. The additional axial term \(w_{AxRed}\) is the axial reduction term from blade bending.

![Figure 3.3: Undeflected blade element offsets](image)

Similar to the angular deflections (Eq. 3.3 and Fig. 2.5), the above deflections in the \(j\)
system are:

\[ u^{B1}(r) = \Psi_{1,1}^{B1}(r) \Phi_1(r) * q_{B1F1} + \Psi_{1,2}^{B1}(r) \Phi_1(r) * q_{B1F2} \]
\[ + \Psi_{1,3}^{B1}(r) \Phi_1(r) * q_{B1E1} + \Psi_{1,4}^{B1}(r) \Phi_1(r) * q_{B1T1} \]

\[ v^{B1}(r) = \Psi_{2,1}^{B1}(r) \Phi_2(r) * q_{B1F1} + \Psi_{2,2}^{B1}(r) \Phi_2(r) * q_{B1F2} \]
\[ + \Psi_{2,3}^{B1}(r) \Phi_2(r) * q_{B1E1} + \Psi_{2,4}^{B1}(r) \Phi_2(r) * q_{B1T1} \]

\[ w^{B1}(r) = \Psi_{3,1}^{B1}(r) \Phi_3(r) * q_{B1F1} + \Psi_{3,2}^{B1}(r) \Phi_3(r) * q_{B1F2} \]
\[ + \Psi_{3,3}^{B1}(r) \Phi_3(r) * q_{B1E1} + \Psi_{3,4}^{B1}(r) \Phi_3(r) * q_{B1T1} \]

with the values of the interpolation functions (Φ) again equal to one.

The axial reduction term is more complex. The infinitesimal change in element length (see Fig. 2.3) is:

\[ dw_{AxRed} = ds - dl_{AxRed} \] (3.6)

where \( dl_{AxRed} \) is the displaced length due to axial reduction, which is:

\[ dl_{AxRed} = \left( (ds)^2 + \left( \frac{\partial u}{\partial s} ds \right)^2 + \left( \frac{\partial v}{\partial s} ds \right)^2 \right)^{1/2} \] (3.7)

The binomial expansion of Eq. 3.7 with the first two terms is:

\[ dl_{AxRed} \approx ds \left[ 1 + \frac{1}{2} \left( \frac{\partial u}{\partial s} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial s} \right)^2 \right] \] (3.8)

From Eq. 3.6, the infinitesimal change in element length is therefore:

\[ dw_{AxRed} \approx -ds \left[ \frac{1}{2} \left( \frac{\partial u}{\partial s} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial s} \right)^2 \right] \] (3.9)
The total change in element length is:

$$w_{AxRed} = \int_0^l dw_{AxRed}$$

$$= -\int_0^l \left[ \frac{1}{2} \left( \frac{\partial u}{\partial s} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial s} \right)^2 \right] ds$$ (3.10)

This equations resembles the terms in the strain energy relationship (Eq. 2.14), without the axial force term. In the same manner as Eqs. 2.30 and 2.31:

$$[K_{AxRede}] = \begin{bmatrix} [K_{AxReda}] & [K_{AxRedb}] \\ \hline \hline \end{bmatrix}, \quad (3.11)$$

and:

$$[K_{AxReda}] = \int_0^l [N'_1]^T[N'_1] ds, \quad [K_{AxRedb}] = \int_0^l [N'_2]^T[N'_2] ds. \quad (3.12)$$

Appendix A lists the components of $K_{AxRede}$. With the eigenvector matrix:

$$[\Psi] = \{\{\Psi_1\}, \{\Psi_2\}, \{\Psi_3\}, \{\Psi_4\}\}$$

and the vector of degree of freedom displacements:

$$\{q\} = \begin{bmatrix} q_{B1F1} \\ q_{B1F2} \\ q_{B1E1} \\ q_{B1T1} \end{bmatrix}$$

the deflection due to axial reduction from Eq. 3.10 is:

$$w_{AxRed} = -\frac{1}{2} [[\Psi][\{q\}]^T[K_{AxRede}][[\Psi][\{q\}]]$$
where $[K_{AxRed}]$ is the global axial reduction matrix. For a particular node’s axial reduction, the eigenvectors and the axial reduction matrix include the element degrees of freedom at the node in addition the degrees of freedom of all the inboard nodes. Using the transpose property:

$$[[A][B]]^T = [B]^T[A]^T$$

the above relation becomes

$$w_{AxRed} = -\frac{1}{2}\{q\}^T[\Psi]^T[K_{AxRed}][[\Psi]\{q\}]$$

Expanding:

$$w_{AxRed} =$$

$$-\frac{1}{2}\{\psi_1\}q_{BF1} + \{\psi_2\}q_{BF2} + \{\psi_3\}q_{BE1} + \{\psi_4\}q_{BT1}\}^T$$

$$[K_{AxRed}]\{\{\psi_1\}q_{BF1} + \{\psi_2\}q_{BF2} + \{\psi_3\}q_{BE1} + \{\psi_4\}q_{BT1}\}$$

Multiplying the terms through:

$$w_{AxRed} =$$

$$-\frac{1}{2}\left\{\{\psi_1\}^T[K_{AxRed}]\{\psi_1\}q_{BF1}^2 + \{\psi_2\}^T[K_{AxRed}]\{\psi_2\}q_{BF2}^2 +$$

$$\{\psi_3\}^T[K_{AxRed}]\{\psi_3\}q_{BE1}^2 + \{\psi_4\}^T[K_{AxRed}]\{\psi_4\}q_{BT1}^2 +$$

$$2\{\psi_1\}^T[K_{AxRed}]\{\psi_2\}q_{BF1}q_{BF2} + 2\{\psi_1\}^T[K_{AxRed}]\{\psi_3\}q_{BF1}q_{BE1} +$$

$$2\{\psi_1\}^T[K_{AxRed}]\{\psi_4\}q_{BF1}q_{BT1} + 2\{\psi_2\}^T[K_{AxRed}]\{\psi_3\}q_{BF2}q_{BE1} +$$

$$2\{\psi_2\}^T[K_{AxRed}]\{\psi_4\}q_{BF2}q_{BT1} + 2\{\psi_3\}^T[K_{AxRed}]\{\psi_4\}q_{BE1}q_{BT1}\right\}$$
or:

\[ w_{Ax,Red} = \]
\[-\frac{1}{2} \{ S_{11}q_{B1F1}^2 + S_{22}q_{B1F2}^2 + S_{33}q_{B1E1}^2 + S_{44}q_{B1T1}^2 + \]
\[ 2S_{12}q_{B1F1}q_{B1F2} + 2S_{13}q_{B1F1}q_{B1E1} + 2S_{14}q_{B1F1}q_{B1T1} + \]
\[ 2S_{23}q_{B1F2}q_{B1E1} + 2S_{24}q_{B1F2}q_{B1T1} + 2S_{34}q_{B1E1}q_{B1T1} \}\]

(3.13)

where:

\[ S_{ij} = \{ \Psi_i \}^T K_{Ax,Red} \{ \Psi_j \}; \quad i, j = 1, 2, 3, 4 \]

### 3.3.2 Velocities

The angular velocity of the blade 1 element body (M1) in the inertial frame is:

\[ E\omega^{M1}(r) = E\omega^H + H\omega^{M1}(r) \]  

(3.14)

The analysis uses this angular velocity with the aerodynamic moments in the development of the generalized active forces and for the blade twist in the generalized inertia force. The simulation does not calculate the flap and edge rotation generalized inertia forces, which is consistent with the Bernoulli hypothesis.

The angular velocity \( E\omega^H \) is the angular velocity of the hub in the inertial frame and the angular velocity of the element body in the hub frame is:

\[ H\omega^{M1}(r) = \{ \dot{\theta}^{B1}_{jx}(r), \dot{\theta}^{B1}_{jy}(r), \dot{\theta}^{B1}_{jz}(r) \} \begin{bmatrix} J_1^{Bi} \\ J_2^{Bi} \\ J_3^{Bi} \end{bmatrix} \]  

(3.15)

The time derivatives of the element angular deflections come from taking the time deriva-
The velocity of a blade element node in the inertial frame is:

\[
\dot{\theta}^{BI}(r) = \Psi_{4,1}^{BI}(r) \Phi_4(r) \dot{q}_{BIF1} + \Psi_{4,2}^{BI}(r) \Phi_4(r) \dot{q}_{BIF2} + \Psi_{4,3}^{BI}(r) \Phi_4(r) \dot{q}_{BIE1} + \Psi_{4,4}^{BI}(r) \Phi_4(r) \dot{q}_{BIT1}
\]

\[
\dot{\theta}^{BI}(r) = \Psi_{5,1}^{BI}(r) \Phi_5(r) \dot{q}_{BIF1} + \Psi_{5,2}^{BI}(r) \Phi_5(r) \dot{q}_{BIF2} + \Psi_{5,3}^{BI}(r) \Phi_5(r) \dot{q}_{BIE1} + \Psi_{5,4}^{BI}(r) \Phi_5(r) \dot{q}_{BIT1}
\]

\[
\dot{\theta}^{BI}(r) = \Psi_{6,1}^{BI}(r) \Phi_6(r) \dot{q}_{BIF1} + \Psi_{6,2}^{BI}(r) \Phi_6(r) \dot{q}_{BIF2} + \Psi_{6,3}^{BI}(r) \Phi_6(r) \dot{q}_{BIE1} + \Psi_{6,4}^{BI}(r) \Phi_6(r) \dot{q}_{BIT1}
\]

The velocity in the hub frame comes from the time derivative of Eq. 3.4:

\[
E^vS^1(r) = E^vQ + H^vS^1(r) + E^\omega \times r^{QS^1}(r)
\]

The velocity in the hub frame comes from the time derivative of Eq. 3.4:

\[
H^vS^1(r) = \frac{H^dr^{QS^1}(r)}{dt}
\]

\[
= \{ \dot{u}^{BI}(r), \dot{v}^{BI}(r), \dot{w}^{BI}(r) + \dot{w}^{BI}_{AxRed}(r) \} \begin{pmatrix}
\dot{j}_{1B}^I \\
\dot{j}_{2B}^I \\
\dot{j}_{3B}^I
\end{pmatrix}
\]

Pitch dynamics are not included in FAST, therefore the derivative of the \( j \) unit vectors are zero. The time derivatives of the element deflections come from the time derivatives of Eq.
3.5:

\[ \ddot{u}^{B1}(r) = \Psi_{1,1}^{B1}(r)\Phi_1(r)\dot{q}_{B1F1} + \Psi_{1,2}^{B1}(r)\Phi_1(r)\dot{q}_{B1F2} + \Psi_{1,3}^{B1}(r)\Phi_1(r)\dot{q}_{B1E1} + \Psi_{1,4}^{B1}(r)\Phi_1(r)\dot{q}_{B1T1} \]

\[ \ddot{v}^{B1}(r) = \Psi_{2,1}^{B1}(r)\Phi_2(r)\dot{q}_{B1F1} + \Psi_{2,2}^{B1}(r)\Phi_2(r)\dot{q}_{B1F2} + \Psi_{2,3}^{B1}(r)\Phi_2(r)\dot{q}_{B1E1} + \Psi_{2,4}^{B1}(r)\Phi_2(r)\dot{q}_{B1T1} \]

\[ \ddot{w}^{B1}(r) = \Psi_{3,1}^{B1}(r)\Phi_3(r)\dot{q}_{B1F1} + \Psi_{3,2}^{B1}(r)\Phi_3(r)\dot{q}_{B1F2} + \Psi_{3,3}^{B1}(r)\Phi_3(r)\dot{q}_{B1E1} + \Psi_{3,4}^{B1}(r)\Phi_3(r)\dot{q}_{B1T1} \]

and the time derivative of Eq. 3.13:

\[ \dot{w}_{AxRed} = - \left\{ S_{11}\dot{q}_{B1F1}\dot{q}_{B1F1} + S_{22}\dot{q}_{B1F2}\dot{q}_{B1F2} + S_{33}\dot{q}_{B1E1}\dot{q}_{B1E1} + S_{44}\dot{q}_{B1T1}\dot{q}_{B1T1} + S_{12}(\dot{q}_{B1F1}\dot{q}_{B1F2} + \dot{q}_{B1F1}\dot{q}_{B1F2}) + S_{13}(\dot{q}_{B1F1}\dot{q}_{B1E1} + \dot{q}_{B1F1}\dot{q}_{B1E1}) + S_{14}(\dot{q}_{B1F1}\dot{q}_{B1T1} + \dot{q}_{B1F1}\dot{q}_{B1T1}) + S_{23}(\dot{q}_{B1F2}\dot{q}_{B1E1} + \dot{q}_{B1F2}\dot{q}_{B1E1}) + S_{24}(\dot{q}_{B1F2}\dot{q}_{B1T1} + \dot{q}_{B1F2}\dot{q}_{B1T1}) + S_{34}(\dot{q}_{B1E1}\dot{q}_{B1T1} + \dot{q}_{B1E1}\dot{q}_{B1T1}) \right\} \]

The partial angular velocity, used for Kane’s method [25], comes from the partial velocity derivative of Eq. 3.14:

\[ E_\omega^M (r) = E_\omega^H + H_\omega^M (r) \quad (3.18) \]

with:

\[ H_\omega^M (r) = \left\{ \dot{j}_{1x}^{BI}(r), \dot{j}_{1y}^{BI}(r), \dot{j}_{1z}^{BI}(r) \right\}_r \left\{ ^{j_1}_{J_1} \right\} + \left\{ \dot{j}_{2x}^{BI}(r), \dot{j}_{2y}^{BI}(r), \dot{j}_{2z}^{BI}(r) \right\}_r \left\{ ^{j_2}_{J_2} \right\} + \left\{ \dot{j}_{3x}^{BI}(r), \dot{j}_{3y}^{BI}(r), \dot{j}_{3z}^{BI}(r) \right\}_r \left\{ ^{j_3}_{J_3} \right\} \]
where the $r$ subscript represents the partial velocity derivative. Expanding Eq. 3.18:

$$E^{M_i}_{r}(r) = E^H_{r} + \begin{cases} 
H^{M_i}_{B1F1}(r) & \text{for } r = B1F1 \\
H^{M_i}_{B1F2}(r) & \text{for } r = B1F2 \\
H^{M_i}_{B1E1}(r) & \text{for } r = B1E1 \\
H^{M_i}_{B1T1}(r) & \text{for } r = B1T1 \\
0 & \text{otherwise} 
\end{cases} \quad (3.19)$$

with the additional partial velocities coming from the partial velocity derivative of Eq. 3.15:

$$H^{M_i}_{B1F1}(r) = \{ \Psi_{4,1}^{B1}(r) \Phi_4(r), \Psi_{5,1}^{B1}(r) \Phi_5(r), \Psi_{6,1}^{B1}(r) \Phi_6(r) \}$$

$$H^{M_i}_{B1F2}(r) = \{ \Psi_{4,2}^{B1}(r) \Phi_4(r), \Psi_{5,2}^{B1}(r) \Phi_5(r), \Psi_{6,2}^{B1}(r) \Phi_6(r) \}$$

$$H^{M_i}_{B1E1}(r) = \{ \Psi_{4,3}^{B1}(r) \Phi_4(r), \Psi_{5,3}^{B1}(r) \Phi_5(r), \Psi_{6,3}^{B1}(r) \Phi_6(r) \}$$

$$H^{M_i}_{B1T1}(r) = \{ \Psi_{4,4}^{B1}(r) \Phi_4(r), \Psi_{5,4}^{B1}(r) \Phi_5(r), \Psi_{6,4}^{B1}(r) \Phi_6(r) \}$$
The partial linear velocity comes from the partial velocity derivative of Eq. 3.16:

\[ E \mathbf{v}_r^{S1}(r) = E \mathbf{v}_r^{Q} + H \mathbf{v}_r^{S1}(r) + E \mathbf{\omega}_r^H \times r^{QS1}(r) \]  

(3.20)

with:

\[ H \mathbf{v}_r^{S1}(r) = \{ \dot{u}^{B1}(r), \dot{v}^{B1}(r), \dot{w}^{B1}(r) \}_r \begin{pmatrix} \dot{j}_1^{B1} \\ \dot{j}_2^{B1} \\ \dot{j}_3^{B1} \end{pmatrix} \]

Expanding Eq. 3.20:

\[ E \mathbf{v}_r^{S1}(r) = E \mathbf{v}_r^{Q} + \begin{cases} 
E \mathbf{\omega}_r^H \times r^{QS1}(r) & \text{for } r = 4, 5, \ldots, 14 \\
H \mathbf{v}_{B1F1}^{S1}(r) & \text{for } r = B1F1 \\
H \mathbf{v}_{B1F2}^{S1}(r) & \text{for } r = B1F2 \\
H \mathbf{v}_{B1E1}^{S1}(r) & \text{for } r = B1E1 \\
H \mathbf{v}_{B1T1}^{S1}(r) & \text{for } r = B1T1 \\
E \mathbf{\omega}_{Teet}^H \times r^{QS1}(r) & \text{for } r = Teet \\
0 & \text{otherwise}
\end{cases} \]  

(3.21)
with the additional partial velocities coming from the partial velocity derivative of Eq. 3.17:

\[
H v_{B1Fi}^S(r) = \Psi_{1,1}^B(r) \Phi_1(r) j_1^B + \Psi_{2,1}^B(r) \Phi_2(r) j_2^B + (\Psi_{3,1}^B(r) \Phi_3(r) - \{S_{11} q_{B1Fi} + S_{12} q_{B1F2} + S_{13} q_{BIE1} + S_{14} q_{BIT1}\}) j_3^B
\]

\[
H v_{B1F2}^S(r) = \Psi_{1,2}^B(r) \Phi_1(r) j_1^B + \Psi_{2,2}^B(r) \Phi_2(r) j_2^B + (\Psi_{3,2}^B(r) \Phi_3(r) - \{S_{22} q_{B1F2} + S_{12} q_{B1Fi} + S_{23} q_{BIE1} + S_{24} q_{BIT1}\}) j_3^B
\]

\[
H v_{B1E1}^S(r) = \Psi_{1,3}^B(r) \Phi_1(r) j_1^B + \Psi_{2,3}^B(r) \Phi_2(r) j_2^B + (\Psi_{3,3}^B(r) \Phi_3(r) - \{S_{33} q_{B1E1} + S_{13} q_{B1Fi} + S_{23} q_{B1F2} + S_{34} q_{BIT1}\}) j_3^B
\]

\[
H v_{B1T1}^S(r) = \Psi_{1,4}^B(r) \Phi_1(r) j_1^B + \Psi_{2,4}^B(r) \Phi_2(r) j_2^B + (\Psi_{3,4}^B(r) \Phi_3(r) - \{S_{44} q_{B1T1} + S_{14} q_{B1Fi} + S_{24} q_{B1F2} + S_{34} q_{BIE1}\}) j_3^B
\]

### 3.3.3 Accelerations

In Kane’s method [25], angular accelerations are:

\[
E \alpha_{N_i}(\dot{q}, \ddot{q}, q, t) = \sum_{r=1}^{\text{NDOF}} E \omega_{r}^{N_i}(q, t) \ddot{q}_r + \sum_{r=1}^{\text{NDOF}} \frac{d}{dt} \left( E \omega_{r}^{N_i}(q, t) \right) \dot{q}_r + \frac{d}{dt} \left( E \omega_{r}^{N_i}(q, t) \right)
\]

for each body \( N_i \) in the system.

The term required for the angular acceleration comes from the time derivative of the partial angular velocity (Eq. 3.19):
\[ \frac{d}{dt} \left[ E \omega_r^{M1} \right] = \frac{d}{dt} \left( E \omega_r^H \right) + \]

\[
\begin{cases} 
E \omega_r^H \times E \omega_{B1F1}^{M1} (r) & \text{for } r = B1F1 \\
E \omega_r^H \times E \omega_{B1F2}^{M1} (r) & \text{for } r = B1F2 \\
E \omega_r^H \times E \omega_{B1E1}^{M1} (r) & \text{for } r = B1E1 \\
E \omega_r^H \times E \omega_{B1T1}^{M1} (r) & \text{for } r = B1T1 \\
0 & \text{otherwise}
\end{cases}
\]

Linear accelerations are:

\[
E a^{X_i} (\ddot{q}, \dot{q}, q, t) = \sum_{r=1}^{\text{NDOF}} E v_r^{X_i} (q, t) \ddot{q}_r + \sum_{r=1}^{\text{NDOF}} \frac{d}{dt} (E v_r^{X_i} (q, t)) \dot{q}_r + \frac{d}{dt} (E v_r^{X_i} (q, t)) \]  \quad (3.22)

for each point \(X_i\) in the system.

The term required for the linear acceleration is given by the time derivative of the partial
velocity (Eq. 3.21):

\[ \frac{d}{dt} \left[ E_\nu^{SL}(r) \right] = \frac{d}{dt} \left( E_\nu^Q \right) + \begin{cases} E_\omega^H \times \left[ H_\nu^{SL}(r) + E_\omega^H \times r^{QS1}(r) \right] & \text{for } r = 4,5,6 \\ \frac{d}{dt} \left( E_\omega^H \right) \times r^{QS1}(r) + E_\omega^H \times \left[ H_\nu^{SL}(r) + E_\omega^H \times r^{QS1}(r) \right] & \text{for } r = 7,8, \ldots, 14 \\ -[S_{11}\dot q^{B1F1} + S_{12}\dot q^{B1F2} + S_{13}\dot q^{B1E1}] \\ S_{14}\dot q^{B1TI}]_3^{B1} + E_\omega^H \times H_\nu^{SL}(r) & \text{for } r = B1F1 \\ -[S_{22}\dot q^{B1F2} + S_{12}\dot q^{B1F1} + S_{23}\dot q^{B1E1}] \\ S_{24}\dot q^{B1TI}]_3^{B1} + E_\omega^H \times H_\nu^{SL}(r) & \text{for } r = B1F2 \\ -[S_{33}\dot q^{B1E1} + S_{13}\dot q^{B1F1} + S_{23}\dot q^{B1F2}] \\ S_{34}\dot q^{B1E1}]_3^{B1} + E_\omega^H \times H_\nu^{SL}(r) & \text{for } r = B1E1 \\ -[S_{44}\dot q^{B1TI} + S_{14}\dot q^{B1F1} + S_{24}\dot q^{B1F2}] \\ S_{34}\dot q^{B1E1}]_3^{B1} + E_\omega^H \times H_\nu^{SL}(r) & \text{for } r = B1T1 \\ \frac{d}{dt} \left( E_\omega^H \right) \times r^{QS1}(r) \\ E_\omega^H \times \left[ H_\nu^{SL}(r) + E_\omega^H \times r^{QS1}(r) \right] & \text{for } r = \text{Teet} \\ 0 & \text{otherwise} \end{cases} \]

3.4 Kinetics

Kane’s equations of motion [25] for a simple holonomic system are:

\[ F_r + F_r^* = 0 \ (r = 1, 2, \ldots, \text{NDOF}) \]
where the generalized active forces are:

\[ F_r = \sum_{i=1}^{u} E v_r^{X_i} \cdot F^{X_i} + E \omega_r^{N_i} \cdot M^{N_i} \]

and the generalized inertia forces are:

\[ F^*_r = \sum_{i=1}^{u} E v_r^{X_i} \cdot (-m_r^{N_i} E \alpha^{X_i}) + E \omega_r^{N_i} \cdot (-E \vec{H}^{N_i}) \]

where the forces \( F^{X_i} \) are applied at the center of mass point \( X_i \) for each rigid body \( N_i \). The time derivative of the angular momentum of rigid body \( N_i \) about its center of mass \( X_i \) in the inertial reference frame \( E \) is:

\[ E \dot{H}^{N_i} = \vec{I}^{N_i} \cdot E \alpha^{N_i} + E \omega^{N_i} \times \vec{I}^{N_i} \cdot E \omega^{N_i} \quad (3.23) \]

where \( \vec{I}^{N_i} \) is the inertia dyadic of the body.

FAST uses Kane’s equation of motion in matrix form, as in:

\[ [C(q, t)] \{ \ddot{q} \} + \{ f(\dot{q}, q, t) \} = \{ 0 \} \]

or:

\[ [C(q, t)] \{ \ddot{q} \} = -\{ f(\dot{q}, q, t) \} \]

### 3.4.1 Generalized Inertia Forces

For the generalized inertia forces, the program determines the contribution from each blade element and the tip-node body. In the ensuing analysis, the author has only developed the equations for blade 1; there are similar equations for the other blades.
The generalized inertia force for a blade 1 element that includes torsion is:

\[
F_r^{M1}(r) = -\text{NodeMass}(r)E v_r^{S1}(r) \cdot E a_r^{S1}(r) dr \\
+ E \omega_r^{M1}(r) \cdot \left( E \dot{H}^{M1}(r) \right)
\]

where \text{NodeMass} is:

\[
\text{NodeMass}(r) = \text{AdjBlMs}^{B1} \cdot \text{BMassDen}^{B1}(r) \cdot \text{DRNodes}^{B1}(r)
\]

where \text{AdjBlMs} is the blade mass adjustment factor in the blade file, \text{BMassDen} is the blade mass density per unit length for the analysis node, and \text{DRNodes} is the element length. Note that, as in \text{FAST2ADAMS.f90} pre-processor, the node mass was not adjusted for additional length from blade sweep and curvature. Therefore, there is an error that grows with sweep angle. This can be countered by multiplying the blade mass density with:

\[
\text{BMassDen}_{new} = \text{BMassDen}_{old} \cdot \frac{\text{Actual element length}}{\text{DRNodes}}
\]

With the addition of twist, the model includes the generalized inertia force for the twist motion. For the twist motion the blade element dyadic is:

\[
\bar{I}^{M1}(r) = ((\text{InerBFLp}^{B1}(r) + \text{InerBEedg}^{B1}(r)) \cdot DRNodes(r) + \text{SmlLnmbbr})n_3^{B1}(r)n_3^{B1}(r)
\]

The variable \text{InerBFLp} is the flap inertia per unit length. Similarly, \text{InerBEedg} is the edge inertia per unit length. \text{DRNodes} is the element length, and \text{SmlLnmbbr} is a small number (9.999E-4) so that an infinite acceleration is not computed. This inertia is similar to the one computed in \text{FAST2ADAMS.f90}

Using the Eq. 3.24 and previously derived terms, the total generalized inertia force for
blade 1 is:

\[ F^*_{r \mid B1} = - \left[ \sum_{j=1}^{\text{BldNodes}} \text{NodeMass}(j) E \mathbf{v}^{SI}_i(j) \cdot \left\{ \left( \sum_{i=1}^{14} E \mathbf{v}^{SI}_i(j) \dot{q}_i \right) + \left( \sum_{i=1}^{19} \frac{d}{dt} (E \mathbf{v}^{SI}_i(j)) \dot{q}_i \right) \right\} \right] + 
\left\{ \bar{I}^{MI}(j) \cdot \left\{ \left( \sum_{i=4}^{14} E \omega^{MI}_i(j) \dot{q}_i \right) + \left( \sum_{i=16}^{19} E \omega^{MI}_i(j) \dot{q}_i \right) \right\} \right\} + E \omega^{MI}_j \times \bar{I}^{MI}(j) \cdot E \omega^{MI}(j) \right] + \ldots \] 

Continuing with the blade 1 generalized inertia force:
\[ F_r^*|_{B1} = \ldots + \]

\[-m^\text{B1Tip} \, E \mathbf{v}_{r}^{\text{SI}}(\text{BldFlexL}) \cdot \]

\[ \left\{ \left( \sum_{i=1}^{14} E \mathbf{v}_{i}^{\text{SI}}(\text{BldFlexL}) \ddot{q}_i \right) + \left( \sum_{i=16}^{19} E \mathbf{v}_{i}^{\text{SI}}(\text{BldFlexL}) \ddot{q}_i \right) \right\} + \]

\[ E \mathbf{v}_{\text{TeeT}}^{\text{SI}}(\text{BldFlexL}) \ddot{q}_{\text{TeeT}} + \left( \sum_{i=4}^{14} \frac{d}{dt} \left( E \mathbf{v}_{i}^{\text{SI}}(\text{BldFlexL}) \right) \dot{q}_i \right) + \]

\[ \left( \sum_{i=16}^{19} \frac{d}{dt} \left( E \mathbf{v}_{i}^{\text{SI}}(\text{BldFlexL}) \right) \dot{q}_i \right) + \frac{d}{dt} \left( E \mathbf{v}_{\text{TeeT}}^{\text{SI}}(\text{BldFlexL}) \right) \dot{q}_{\text{TeeT}} \right\} - \]

\[ E \omega_{r}^{\text{MI}}(\text{BldFlexL}) \cdot \left\{ \bar{I}_{\text{B1Tip}} \cdot \left\{ \left( \sum_{i=4}^{14} E \omega_{i}^{\text{MI}}(\text{BldFlexL}) \ddot{q}_i \right) + \right. \]

\[ \left( \sum_{i=16}^{19} E \omega_{i}^{\text{MI}}(\text{BldFlexL}) \ddot{q}_i \right) + E \omega_{\text{TeeT}}^{\text{MI}}(\text{BldFlexL}) \ddot{q}_{\text{TeeT}} + \]

\[ \left( \sum_{i=7}^{14} \frac{d}{dt} \left( E \omega_{i}^{\text{MI}}(\text{BldFlexL}) \right) \dot{q}_i \right) + \left( \sum_{i=16}^{19} \frac{d}{dt} \left( E \omega_{i}^{\text{MI}}(\text{BldFlexL}) \right) \dot{q}_i \right) + \]

\[ \frac{d}{dt} \left( E \omega_{\text{TeeT}}^{\text{MI}}(\text{BldFlexL}) \right) \dot{q}_{\text{TeeT}} \right\} + E \omega_{\text{MI}}(\text{BldFlexL}) \times \]

\[ \bar{I}_{\text{B1Tip}} \cdot E \omega_{\text{MI}}(\text{BldFlexL}) \right\} \]

Rearranging the blade 1 generalized inertia forces (for a two-bladed turbine) into matrix
form:

\[
[C(q, t)]_{B1}(Row, Col) =
\left[ \sum_{j=1}^{\text{BldNodes}} \text{NodeMass}(j) E^{\text{SI}}_{\text{Row}}(j) \cdot E^{\text{SI}}_{\text{Col}}(j) +
E^{\text{MI}}_{\text{Row}}(j) \cdot \bar{I}^{\text{MI}}(j) \cdot E^{\text{MI}}_{\text{Col}}(j) \right] +

m^{\text{B1Tip}} E^{\text{SI}}_{\text{Row}}(\text{BldFlexL}) \cdot E^{\text{SI}}_{\text{Col}}(\text{BldFlexL}) +

E^{\text{MI}}_{\text{Row}}(\text{BldFlexL}) \cdot \bar{I}^{\text{B1Tip}} \cdot E^{\text{MI}}_{\text{Col}}(\text{BldFlexL})
\]

\((Row, Col = 1, 2, \ldots, 14, 16, 17, 18, 19; 24)\)
\[ \{ -f(\dot{q}, q, t) \} \big|_{B1}(\text{Row}) = \]

\[ - \sum_{j=1}^{\text{BladNodes}} \text{NodeMass}(j) E \nu_{\text{Row}}^{\text{SI}}(j) \cdot \left\{ \left( \sum_{i=4}^{14} \frac{d}{dt} (E \nu_{i}^{\text{SI}}(j)) \dot{q}_{i} \right) + \left( \sum_{i=16}^{19} \frac{d}{dt} (E \nu_{i}^{\text{SI}}(j)) \dot{q}_{i} \right) \right\} + \]

\[ E \omega_{\text{Row}}^{\text{MI}}(j) \cdot \left( \left( \sum_{i=4}^{14} \frac{d}{dt} (E \omega_{i}^{\text{MI}}(j)) \dot{q}_{i} \right) + \left( \sum_{i=16}^{19} \frac{d}{dt} (E \omega_{i}^{\text{MI}}(j)) \dot{q}_{i} \right) \right) + \frac{d}{dt} (E \omega_{\text{TeeT}}^{\text{MI}}(j)) \dot{q}_{\text{TeeT}} \right\} - \]

\[ m_{B1\text{Tip}} E \nu_{\text{Row}}^{\text{SI}}(\text{BldFlexL}) \cdot \left\{ \left( \sum_{i=4}^{14} \frac{d}{dt} (E \nu_{i}^{\text{SI}}(\text{BldFlexL})) \dot{q}_{i} \right) + \left( \sum_{i=16}^{19} \frac{d}{dt} (E \nu_{i}^{\text{SI}}(\text{BldFlexL})) \dot{q}_{i} \right) \right\} - \]

\[ E \omega_{\text{Row}}^{\text{MI}}(\text{BldFlexL}) \cdot \left( \left( \sum_{i=7}^{14} \frac{d}{dt} (E \omega_{i}^{\text{MI}}(\text{BldFlexL})) \dot{q}_{i} \right) + \left( \sum_{i=17}^{19} \frac{d}{dt} (E \omega_{i}^{\text{MI}}(\text{BldFlexL})) \dot{q}_{i} \right) \right) + \]

\[ \left( \sum_{i=16}^{19} \frac{d}{dt} (E \omega_{i}^{\text{MI}}(\text{BldFlexL})) \dot{q}_{i} \right) + \frac{d}{dt} (E \omega_{\text{TeeT}}^{\text{MI}}(\text{BldFlexL})) \dot{q}_{\text{TeeT}} \right\} + \]

\[ E \omega^{\text{MI}}(\text{BldFlexL}) \times \bar{I}^{\text{B1Tip}} \cdot E \omega^{\text{MI}}(\text{BldFlexL}) \right\} \]

\((\text{Row} = 1, 2, \ldots, 14; 16, 17, 18, 19; 24)\)

### 3.4.2 Generalized Active Forces

The blade 1 generalized active forces include the gravitational forces, the aerodynamic forces, the elastic forces, and the damping forces, as in:

\[ F_{r} = F_{r}\big|_{\text{GravB1}} + F_{r}\big|_{\text{AeroB1}} + F_{r}\big|_{\text{ElasticB1}} + F_{r}\big|_{\text{DampB1}} \]
**Blade Gravity Forces**

The generalized forces due to gravity are in the same form as the current version of FAST, and is:

\[
F_{r|GravB1} = - \int_{0}^{BldFlexL} \mu_{B1}(r) g E v_{r}^{B1}(r) \cdot z_2 dr - m^{B1Tip} g E v_{B1}^{B1}(BldFlexL) \cdot z_2
\]

(subscript \( r = 3, 4, \ldots, 14; 16, 17, 18, 19; 24 \))

where \( g \) is the gravitational acceleration and \( z_2 \) is the tower base unit vector that points vertical. In matrix form:

\[
[C(q, t)]_{GravB1} = 0
\]

and:

\[
\{-f(\dot{q}, q, t)\}_{GravB1}(Row) = - \int_{0}^{BldFlexL} \mu_{B1}(r) g E v_{Row}^{B1}(r) \cdot z_2 dr - m^{B1Tip} g E v_{Row}^{B1}(BldFlexL) \cdot z_2
\]

\( (Row = 3, 4, \ldots, 14; 16, 17, 18, 19; 24) \)

**Blade Aerodynamic Forces**

The application of the aerodynamic forces is the same as in FAST, with:

\[
F_{r|AeroB1} = \int_{0}^{BldFlexL} \left[ E v_{r}^{S1}(r) \cdot F_{AeroB1}^{S1}(r) + E \omega_{r}^{M1}(r) \cdot M_{AeroB1}^{M1}(r) \right] dr + E v_{r}^{S1}(BldFlexL) \cdot F_{TipDragB1}^{S1}(BldFlexL)
\]

(subscript \( r = 1, 2, \ldots, 14; 16, 17, 18, 19; 24 \))
where \( F^{S1}_{AeroB1}(r) \) and \( M^{M1}_{AeroB1}(r) \) are aerodynamic forces and pitching moments applied to point \( S1 \) on blade 1 expressed per unit span. The pitching moment can include moments due to airfoil moment coefficients \( (C_M) \) and moments due to offsets of the aerodynamic center to the element center of mass.

In matrix form the generalized active aerodynamic forces are:

\[
[C(q, t)]_{AeroB1} = 0
\]

and:

\[
\{ - f(\dot{q}, q.t) \}_{AeroB1}(Row) = \int_0^{BldFlexL} \left[ E_{Row}(r) \cdot F^{S1}_{AeroB1}(r) + E_{\omega}^{M1}_{Row}(r) \cdot M^{M1}_{AeroB1}(r) \right] dr + E_{\mathbf{v}}^{S1}_{Row}(BldFlexL) \cdot F^{S1}_{TipDragB1}(BldFlexL)
\]

\((Row = 1, 2, \ldots, 14; 16, 17, 18, 19; 24)\)

As stated in Chapter 1, FAST uses the AeroDyn subroutines to compute the aerodynamic forces. However, there is no accounting for blade sweep in AeroDyn. The author therefore made corrections to the two-dimensional airfoil tables that are used to compute the forces.

The aerodynamic correction for sweep takes into account that the pressure forces on the blade are dependent on the relative velocity normal to the rotor leading edge. This is the same analogy used for fixed wing aircraft (see Hoerner [49]). This assumption is not used by researchers in the rotorcraft community; however, researchers with the STAR program studied the use of this assumption. They used a lifting-surface wake code originally developed for helicopter rotors and modified for wind turbines [50] to study the effect of sweep on a model wind turbine. It was found that the spanwise loading closely matched the typical modification to the lift coefficient for sweep:
\[ C_{L\Delta} = C_L \cos^2 \Lambda \]  

(3.25)

where \( \Lambda \) is the angle between the relative velocity and the chord line of the blade section. The drag coefficient was not altered for sweep. The results showed the assumption was acceptable to about 20\(^\circ\) of sweep. For the STAR analysis, however, Adams provides relative velocities already normal to the leading edge to AeroDyn. The author did not modify the lift coefficient; however, the drag coefficient was:

\[ C_{D\Lambda} = \frac{C_D}{\cos^2 \Lambda} \]  

(3.26)

Figure 3.4 shows the angle \( \Lambda \) for the swept blade, assuming no out-of-plane curvature (RefAxis\(xb\)). The sweep angle \( \Lambda \) is the angle between the \( \Omega \times r \) velocity vector and the perpendicular to the reference axis. The formula for \( \Lambda \) is:

\[ \Lambda = \gamma - \beta \]

where \( \beta \) is the angle between the pitch axis and radial position vector, and \( \gamma \) is:

\[ \gamma = \chi - \alpha \]

where \( \alpha \) is the pre-sweep which is machined at the root face the balance the root pitching moment and \( \chi \) is the blade sweep angle. The formula \( \chi \) is the arctangent of the derivative of the formula for the STAR-blade local sweep displacement of the reference axis (RefAxis\(yb\)):

\[ \text{RefAxis}\(yb\) = Tip Sweep (distance) \times \left( \frac{\text{Blade Station} - \text{Start of sweep}}{\text{Blade Length} - \text{Start of sweep}} \right)^{\text{Sweep Exponent}} \]  

(3.27)

for blade stations outboard of the sweep onset station.
Figure 3.4: Sweep angle for swept blade
In addition to the lifting surface studies, van Dam and Saephan built a computation fluid dynamics (CFD) model of the STAR [51] rotor, which showed good agreement to the BEM corrected for sweep as above. The researchers therefore had further confidence in using this assumption for sweep.

The CurveFAST modification duplicates the current method used in the Adams model, by supplying the relative velocities to AeroDyn in the plane perpendicular to the leading edge, and the drag coefficients in the two-dimensional airfoil tables are adjusted according to Eq. 3.26.

**Blade Elastic Forces**

For the blade elastic forces, the model uses the results from the blade finite element analysis. From the FAST kinetics development [23] the Blade 1 generalized elastic force is:

\[ F_{l}^{\text{ElasticB1}} = -\frac{\partial V^{\text{BI}}}{\partial q_{r}} \]  

(3.28)

The model approximates the blade elastic potential energy by using four mode shapes, as in:

\[ V = \frac{1}{2} q_{B1F1}^{2} \{\psi_{1}\}^{T} \{K_{e}\} \{\psi_{1}\} + \frac{1}{2} q_{B1F2}^{2} \{\psi_{2}\}^{T} \{K_{e}\} \{\psi_{2}\} + \frac{1}{2} q_{B1E1}^{2} \{\psi_{3}\}^{T} \{K_{e}\} \{\psi_{3}\} + \frac{1}{2} q_{B1T1}^{2} \{\psi_{4}\}^{T} \{K_{e}\} \{\psi_{4}\} \]  

(3.29)

where for example \{\psi_{1}\} is the first mode eigenvector and \{K_{e}\} is the elastic stiffness matrix for the non-rotating blade. The eigenvectors and stiffness matrix are referenced to the pitched \(j\) system. CurveFAST internally calculates the elastic stiffness matrix to reduce the truncation error in the matrix multiplications. The derivation of this matrix is in Chapter
2. The eigenvectors are included at the end of blade data input file.

Substituting Eq. 3.29 into Eq. 3.28:

\[
F_r|_{\text{ElasticB1}} = \begin{cases}
-q_{B1F1} \{\psi_1\}^T \{K_e\} \{\psi_1\} & \text{for } r = B1F1 \\
-q_{B1F2} \{\psi_2\}^T \{K_e\} \{\psi_2\} & \text{for } r = B1F2 \\
-q_{B1E1} \{\psi_3\}^T \{K_e\} \{\psi_3\} & \text{for } r = B1E1 \\
-q_{B1T1} \{\psi_4\}^T \{K_e\} \{\psi_4\} & \text{for } r = B1T1 \\
0 & \text{otherwise}
\end{cases}
\]

The matrix multiplications of \( \psi \) and \( K_e \) are computed at initialization in the subroutine `coeff` in the file `FAST_SML.f90`.

In matrix form, the elastic forces for blade 1 are:

\[
[[C(q, t)]]_{\text{ElasticB1}} = 0
\]

\[
\{-f(\dot{q}, q, t)\}_{\text{ElasticB1}}(16) = -q_{B1F1} \{\psi_1\}^T \{K_e\} \{\psi_1\}
\]

\[
\{-f(\dot{q}, q, t)\}_{\text{ElasticB1}}(17) = -q_{B1E1} \{\psi_3\}^T \{K_e\} \{\psi_3\}
\]

\[
\{-f(\dot{q}, q, t)\}_{\text{ElasticB1}}(18) = -q_{B1F2} \{\psi_2\}^T \{K_e\} \{\psi_2\}
\]

\[
\{-f(\dot{q}, q, t)\}_{\text{ElasticB1}}(19) = -q_{B1T1} \{\psi_4\}^T \{K_e\} \{\psi_4\}
\]

\[
\{-f(\dot{q}, q, t)\}_{\text{ElasticB1}}(\text{other rows}) = 0
\]

As in FAST, blade damping is assumed to be proportional to the stiffness, which is a typical assumption in modal analysis. The damping, or dissipative function in the Lagrangian development from finite element analysis as in Rao [28], Eq. 12.20, is given as:

\[
R = \frac{1}{2} \{\dot{q}\}^T [D] \{\dot{q}\} \quad (3.30)
\]

where \( Q \) is the vector of nodal displacements and \( \{C\} \) is the damping matrix. For a partic-
ular mode (e.g. first blade flap):

\[ \{q\}_{B1F1} = q_{B1F1} \{\psi_1\} \]  \hspace{1cm} (3.31)

Damping proportional to mass and stiffness is (Rao[28], Eq. 12.81):

\[ [D] = a [M] + b [K_e] \]  \hspace{1cm} (3.32)

where \( a \) and \( b \) are constants and \([M]\) is the mass matrix. For a particular mode, the modal damping constant is (Rao[28], Eq. 12.85):

\[ \zeta_i = \frac{a + b\omega^2_i}{2\omega_i} \]

where \( \omega \) is the square root of the mode eigenvalue in rad/s. Setting \( a \) to zero and solving for \( b \):

\[ b = \frac{2\zeta_i}{\omega_i} \]  \hspace{1cm} (3.33)

Substituting Eq. 3.33 into Eq. 3.32 for a particular mode:

\[ [D]_i = \frac{2\zeta_i}{\omega_i} [K_e] \]  \hspace{1cm} (3.34)

Substituting Eqs. 3.31 and 3.34 into Eq. 3.30 the dissipation becomes:

\[ R = \frac{1}{2} q_{B1F1}^2 \frac{2\zeta_1}{\omega_1} \{\psi_1\}^T [K_e] \{\psi_1\} + \]
\[ \frac{1}{2} q_{B1F2}^2 \frac{2\zeta_2}{\omega_2} \{\psi_2\}^T [K_e] \{\psi_2\} + \]
\[ \frac{1}{2} q_{B1E1}^2 \frac{2\zeta_3}{\omega_3} \{\psi_3\}^T [K_e] \{\psi_3\} + \]
\[ \frac{1}{2} q_{B1T1}^2 \frac{2\zeta_4}{\omega_4} \{\psi_4\}^T [K_e] \{\psi_4\} \]  \hspace{1cm} (3.35)
Similar to the elastic force the generalized damping force takes the following form:

\[
F_r|_{\text{DampB1}} = -\frac{\partial R^{BI}}{\partial \dot{q}_r}
\]  

(3.36)

Substituting Eq. 3.36 into Eq. 3.35:

\[
F_r|_{\text{DampB1}} = \begin{cases} 
-\frac{2\zeta_1}{\omega_1} \dot{q}_{B1F1} \{\psi_1\}^T \{K_e\} \{\psi_1\} & \text{for } r = B1F1 \\
-\frac{2\zeta_2}{\omega_2} \dot{q}_{B1F2} \{\psi_2\}^T \{K_e\} \{\psi_2\} & \text{for } r = B1F2 \\
-\frac{2\zeta_3}{\omega_3} \dot{q}_{B1E1} \{\psi_3\}^T \{K_e\} \{\psi_3\} & \text{for } r = B1E1 \\
-\frac{2\zeta_4}{\omega_4} \dot{q}_{B1T1} \{\psi_4\}^T \{K_e\} \{\psi_4\} & \text{for } r = B1T1 \\
0 & \text{otherwise}
\end{cases}
\]

In matrix form, the damping forces for blade 1 are:

\[
[[C(q,t)]]_{\text{DampB1}} = 0
\]

\[-f(\dot{q}, q, t)|_{\text{DampB1}}(16) = -\frac{2\zeta_1}{\omega_1} \dot{q}_{B1F1} \{\psi_1\}^T \{K_e\} \{\psi_1\} \]

\[-f(\dot{q}, q, t)|_{\text{DampB1}}(17) = -\frac{2\zeta_3}{\omega_3} \dot{q}_{B1E1} \{\psi_3\}^T \{K_e\} \{\psi_3\} \]

\[-f(\dot{q}, q, t)|_{\text{DampB1}}(18) = -\frac{2\zeta_2}{\omega_2} \dot{q}_{B1F2} \{\psi_2\}^T \{K_e\} \{\psi_2\} \]

\[-f(\dot{q}, q, t)|_{\text{DampB1}}(19) = -\frac{2\zeta_4}{\omega_4} \dot{q}_{B1T1} \{\psi_4\}^T \{K_e\} \{\psi_4\} \]

\[-f(\dot{q}, q, t)|_{\text{DampB1}}(\text{other rows}) = 0 \]
3.5 Blade Loads

3.5.1 Blade Root Loads

FAST constructs the equations of motion with loads that are typical quantities of interest. When the equations of motion are solved, these loads are readily available for output and do not require further computations. For the blades, these loads are the blade root forces and moments. Kane’s method breaks the forces into the following (for a 2-bladed turbine):

\[ F_{X_i}^{\text{Source}}(\ddot{q}, \dot{q}, q, t) = \left( \sum_{r=1}^{24} F_{X_i}^{\text{Source}_r}(q, t) \ddot{q}_r \right) + F_{X_i}^{\text{Source}_t}(\dot{q}, q, t) \]  

(3.37)

where \( F_{X_i}^{\text{Source}_r} \) are the partial forces and \( F_{X_i}^{\text{Source}_t} \) is all the components of \( F_{X_i}^{\text{Source}} \) not of this form. For the moments:

\[ M_{N_i}^{\text{Source}}(\ddot{q}, \dot{q}, q, t) = \left( \sum_{r=1}^{24} M_{N_i}^{\text{Source}_r}(q, t) \ddot{q}_r \right) + M_{N_i}^{\text{Source}_t}(\dot{q}, q, t) \]  

(3.38)

where \( M_{N_i}^{\text{Source}_r} \) are the partial moments and \( M_{N_i}^{\text{Source}_t} \) are all the components of \( M_{N_i}^{\text{Source}} \) not of this form.

Blade 1’s generalized active force in terms of the loads acting on the hub center of mass (point \( C \)) is:

\[ F_r|_{B_1} = E \nu_r \cdot F^C_{B_1} + E \omega_r \cdot M^H_{B_1} \]  

\((r = 1, 2, \ldots, 24)\)

Because the hub is rigid, the forces on the hub’s center of mass are related to the blade root loads \( F_{B_1}^{S1}(0) \) and \( M_{B_1}^{H}(0) \) by:

\[ F_{B_1}^{S1}(0) = F^C_{B_1} \]

and:

\[ M_{B_1}^{H} = M_{B_1}^{H}(0) + r^{CS1}(0) \times F_{B_1}^{S1}(0) \]
or:
\[ M_{BI}^H = M_{BI}^H(0) + [r^{QS1}(0) - r^{QC}] \times F_{BI}^{S1}(0) \]

Because of the two-point velocity in Kane and Levinson [25]:

\[ E_v^C = E_v^Q + E_\omega^H \times r^{QC} \]

the generalized active force expands to:

\[ F_{r|BI} = (E_v^Q + E_\omega^H \times r^{QC}) \cdot F_{BI}^{S1}(0) + \\
E_\omega^H \cdot \{ M_{BI}^H(0) + [r^{QS1}(0) - r^{QC}] \times F_{BI}^{S1}(0) \} \]

\[ (r = 1, 2, \ldots, 24) \]

Applying the cyclic permutation law of the scalar triple product:

\[ a \cdot (b \times c) = (a \times b) \cdot c \]

the generalized active force becomes:

\[ F_{r|BI} = E_v^Q \cdot F_{BI}^{S1}(0) + E_\omega^H \cdot \{ r^{QC} \times F_{BI}^{S1}(0) \} + \\
E_\omega^H \cdot \{ M_{BI}^H(0) + [r^{QS1}(0) - r^{QC}] \times F_{BI}^{S1}(0) \} \]

\[ (r = 1, 2, \ldots, 24) \]
which simplifies to:

\[ F_{r|B1} = E\nu^Q_r \cdot F_{B1|SI}^{SI}(0) + E\omega_r^H \cdot \left[ M_{B1}^H(0) + r_{SI} Q(0) \times F_{B1|SI}^{SI}(0) \right] \quad (3.39) \]

\[ (r = 1, 2, \ldots, 24) \]

This generalized active force must produce the same effects as the generalized active and inertia forces from blade 1. Therefore:

\[ F_{r|B1} = F_{r|B1}^* + F_{r|AeroB1} + F_{r|GravB1} + F_{r|ElasticB1} + F_{r|DampB1} \quad (r = 1, 2, \ldots, 24) \]

Because \( E\nu^Q \) and \( E\omega_r^H \) are equal to zero unless \( r = 1, 2, \ldots, 14; Teet \), the blade elastic and damping forces do not contribute to the root loads. Therefore:

\[ F_{r|B1} = F_{r|B1}^* + F_{r|AeroB1} + F_{r|GravB1} \quad (r = 1, 2, \ldots, 14; Teet) \]

Expanding:

\[ F_{r|B1} = \int_0^{BldFlexL} E\nu_r^{SI}(r) \cdot \left[ F_{AeroB1}^{SI}(r) - \mu^{B1}(r) g z_2 - \mu^{B1}(r) E\alpha_r^{SI}(r) \right] dr + \int_0^{BldFlexL} E\omega_r^{MI}(r) \cdot \left[ M_{AeroB1}^{MI}(r) - \bar{I}^{MI}(r) \cdot E\alpha_r^{MI}(r) - E\omega_r^{MI}(r) \times \bar{I}^{MI}(r) \cdot E\alpha_r^{MI}(r) \right] dr + \int_0^{BldFlexL} E\nu_r^{SI}(BldFlexL) \cdot \left[ F_{TipDragB1}^{SI} + m^{B1Tip}(g z_2 + E\alpha_r^{SI}(BldFlexL)) + E\omega_r^{MI}(BldFlexL) \cdot \left[ -\bar{I}^{B1Tip} \cdot E\alpha_r^{MI}(BldFlexL) \right] - E\omega_r^{MI}(BldFlexL) \times \bar{I}^{B1Tip} + E\omega_r^{MI}(BldFlexL) \right] \quad (r = 1, 2, \ldots, 14; Teet) \]
Because:

\[ E \nu^{SI}_r(r) = E \nu^Q_r + H \nu^{SI}_r(r) + E \omega^H_r \times r^{QSI}_r(r) \]

the equation above becomes:

\[
F_r |_{BI} = \int_{0}^{BldFlexL} \left[ E \nu^Q_r + H \nu^{SI}_r(r) \right] \cdot \left[ F^{SI}_{AeroB1}(r) - \mu^{BI}(r) g z_2 - \mu^{BI}(r) E \alpha^{SI}(r) \right] dr + \left[ E \nu^Q_r + H \nu^{SI}_r(BldFlexL) \right] \cdot \left\{ \mathbf{F}_{TipDragBI}^{SI} - m^{B1Tip} [g z_2 + E \alpha^{SI}(BldFlexL)] \right\} + \int_{0}^{BldFlexL} \left[ E \omega^H_r \times r^{QSI}(r) \right] \cdot \left[ F^{SI}_{AeroB1}(r) - \mu^{BI}(r) g z_2 - \mu^{BI}(r) E \alpha^{SI}(r) \right] dr + \left[ E \omega^H_r \times r^{QSI}(r) \right] \cdot \left\{ \mathbf{F}_{TipDragBI}^{SI} - m^{B1Tip} [g z_2 + E \alpha^{SI}(BldFlexL)] \right\} + \int_{0}^{BldFlexL} E \omega^M_1(r) \cdot \left[ \mathbf{M}_{AeroB1}^{M1}(r) - \mathbf{I}^{M1}(r) \cdot E \alpha^{M1}(r) - E \omega^M_1(r) \times \mathbf{I}^{M1}(r) \cdot E \omega^M_1(r) \right] dr + E \omega^M_1(r) (BldFlexL) \cdot \left[ -\mathbf{I}^{B1Tip} \cdot E \alpha^{M1}(BldFlexL) - E \omega^M_1(BldFlexL) \times \mathbf{I}^{B1Tip} \right] .
\]

Because \( H \nu^{SI}_r(r) \) is equal to zero (rigid hub) and \( E \omega^M_1(r) \) is equal to \( E \omega^H_r \) with the con-


straint \((r = 1, 2, \ldots, 14; T_{ee}t)\), the preceding equation simplifies to:

\[
F_r|_{B1} = \\
\int_{BldFlexL}^0 E \mathbf{v}_r^Q(r) \cdot \left[ \mathbf{F}_{AeroB1}^S(r) - \mu^{B1}(r) g z_2 - \mu^{B1}(r) E a^S_1(r) \right] dr + \\
E \mathbf{v}_r^Q \cdot \left\{ \mathbf{F}_{TipDragB1}^S - m^{B1Tip} \left[ g z_2 + E a^S_1(BldFlexL) \right] \right\} + \\
\int_{BldFlexL}^0 \left[ E \mathbf{\omega}_r^H \times r^{QS1}(r) \right] \cdot \left[ \mathbf{F}_{AeroB1}^S(r) - \mu^{B1}(r) g z_2 - \mu^{B1}(r) E a^S_1(r) \right] dr + \left[ E \mathbf{\omega}_r^H \times r^{QS1}(r) \right] \cdot \left\{ \mathbf{F}_{TipDragB1}^S - m^{B1Tip} \left[ g z_2 + E a^S_1(BldFlexL) \right] \right\} + \\
\int_{BldFlexL}^0 E \mathbf{\omega}_r^H \cdot \left[ \mathbf{M}_{AeroB1}^{M1}(r) - \mathbf{I}^{M1}(r) \cdot E a^{M1}(r) - \mathbf{M}_{AeroB1}^{M1}(r) \cdot \mathbf{I}^{M1}(r) \right] dr + E \mathbf{\omega}_r^H \cdot \left[ - \mathbf{I}^{B1Tip} \cdot E a^{M1}(BldFlexL) - E a^{M1}(BldFlexL) \times \mathbf{I}^{B1Tip} \cdot E a^{M1}(BldFlexL) \right] (r = 1, 2, \ldots, 14; T_{ee}t)
\]

With the cyclic permutation law of the scalar triple product, the preceding equation be-
comes:

\[
F_r|_{B1} = \int_{BldFlexL}^0 E \nu^Q(r) \cdot \left[ F_{AeroB1}^{Sl}(r) - \mu^{B1}(r) g z_2 - \mu^{B1}(r) E a^{Sl}(r) \right] dr + \\
E \nu^Q(r) \cdot \left\{ F_{TipDragB1}^{Sl} - m^{B1Tip} [g z_2 + E a^{Sl}(BldFlexL)] \right\} + \\
\int_{BldFlexL}^0 E \omega^H_r \cdot \left\{ r^{QS1}(r) \times \left[ F_{AeroB1}^{Sl}(r) - \mu^{B1}(r) g z_2 - \\
\mu^{B1}(r) E a^{Sl}(r) \right] \right\} dr + E \omega^H_r \cdot \left\{ r^{QS1}(BldFlexL) \times \\
\left\{ F_{TipDragB1}^{Sl} - m^{B1Tip} [g z_2 + E a^{Sl}(BldFlexL)] \right\} \right\} + \\
\int_{BldFlexL}^0 E \omega^H_r \cdot \left[ M_{AeroB1}^{MI}(r) - \bar{I}^{MI}(r) \cdot E a^{MI}(r) - \\
E \omega^H_r \cdot \bar{I}^{MI}(r) \cdot E \omega^MI(r) \right] dr + E \omega^H_r \cdot \\
\left[ -\bar{I}^{B1Tip} \cdot E a^{MI}(BldFlexL) - E \omega^MI(BldFlexL) \times \bar{I}^{B1Tip}, \\
E \omega^MI(BldFlexL) \right] \quad (r = 1, 2, \ldots, 14; Teet)
\]

The root force and moment come from the comparison of the preceding equation with Eq. (3.39):

\[
F_{B1}^{Sl}(0) = \\
\int_{BldFlexL}^0 \left[ F_{AeroB1}^{Sl}(r) - \mu^{B1}(r) g z_2 - \mu^{B1}(r) E a^{Sl}(r) \right] dr + \\
F_{TipDragB1}^{Sl} - m^{B1Tip} [g z_2 + E a^{Sl}(BldFlexL)]
\]
and:

\[
M_{B1}^H(0) + r^{QS1}(0) \times F_{B1}^{SI}(0) = \\
\int_0^{BldFlexL} \left[ M_{AeroB1}^{MI}(r) - \bar{I}^{MI}(r) \cdot E \alpha^{MI}(r) - E \omega^{MI}(r) \times \bar{I}^{MI}(r) \right] \, dr + \int_0^{BldFlexL} r^{QS1}(r) \times \left[ F_{AeroB1}^{SI}(r) - \mu^{B1}(r) g z_2 - \mu^{B1}(r) E \alpha^{SI}(r) \right] \, dr + r^{QS1}(BldFlexL) \times \left\{ F_{TipDragB1}^{SI} - m^{B1Tip} \left[ g z_2 + E \alpha^{SI}(BldFlexL) \right] \right\} + \left[ -\bar{I}^{B1Tip} \right] \cdot E \alpha^{MI}(BldFlexL) - E \omega^{MI}(BldFlexL) \times \bar{I}^{B1Tip} \cdot E \omega^{MI}(BldFlexL)
\]

or:

\[
M_{B1}^H(0) = \\
\int_0^{BldFlexL} \left[ M_{AeroB1}^{MI}(r) - \bar{I}^{MI}(r) \cdot E \alpha^{MI}(r) - E \omega^{MI}(r) \times \bar{I}^{MI}(r) \right] \, dr + \int_0^{BldFlexL} r^{QS1}(r) \times \left[ F_{AeroB1}^{SI}(r) - \mu^{B1}(r) g z_2 - \mu^{B1}(r) E \alpha^{SI}(r) \right] \, dr + m^{B1Tip} \left[ g z_2 + E \alpha^{SI}(BldFlexL) \right] + \left[ -\bar{I}^{B1Tip} \right] \cdot E \alpha^{MI}(BldFlexL) - E \omega^{MI}(BldFlexL) \times \bar{I}^{B1Tip} \cdot E \omega^{MI}(BldFlexL)
\]
\[ M_{B1}^H(0) = \]
\[ \int_0^{BldFlexL} \left[ M_{AeroB1}^M(r) - \bar{I}_{M1}^M(r) \cdot E^M \cdot \alpha^M(r) - E^M \times \bar{I}_{M1}^M(r) \cdot E^M \right] \]  
\[ + \left[ \begin{array}{c}
\int_0^{BldFlexL} \left[ r^QSI(r) \times r^QSI(0) \right] \times \left[ F_{AeroB1}^S(r) - \mu^B1(r) g z_2 - \mu^B1(r) E^S \right] \cdot \bar{I}_{M1}^M(r) \cdot E^S \right] 
+ \left[ F_{TipDragB1}^S - m^{B1Tip} \left[ g z_2 + E^S (BldFlexL) \right] \right] \]  
\[ + E^M \alpha^M (BldFlexL) - E^M \omega^M (BldFlexL) \times \bar{I}^{B1Tip}. \]

Therefore:

\[ F_{B1}^S(0) = \]
\[ \int_0^{BldFlexL} \left\{ F_{AeroB1}^S(r) - \mu^B1(r) \right\} \left\{ g z_2 + \left[ \sum_{i=1}^{14} E^V_i (r) \dot{q}_i \right] + \right\} \]
\[ + \left\{ \sum_{i=16}^{19} E^V_i (r) \dot{q}_i \right\} + E^V_{Teet} (BldFlexL) \dot{q}_{Teet} + \left\{ \sum_{i=4}^{14} \frac{d}{dt} \left( E^V_i (BldFlexL) \right) \right\} + F_{TipDragB1}^S \]
\[ + m^{B1Tip} \left\{ g z_2 + \left[ \sum_{i=1}^{14} E^V_i (BldFlexL) \right] + \right\} \]
\[ + \left\{ \sum_{i=16}^{19} E^V_i (BldFlexL) \right\} + E^V_{Teet} (BldFlexL) + \right\} \]
\[ + \left\{ \sum_{i=4}^{14} \frac{d}{dt} \left( E^V_i (BldFlexL) \right) \right\} + \frac{d}{dt} \left( E^V_{Teet} (BldFlexL) \right) + \right\} \]
and:

\[ M^H_{B1}(0) = \int_0^{BldFlexL} \left[ M^{AeroB1}(r) - \bar{I}^{MI}(r) \cdot \left\{ \sum_{i=4}^{14} E \omega_i^{MI}(r) \ddot{q}_i \right\} + \sum_{i=16}^{19} E \omega_i^{MI}(r) \ddot{q}_i \right] \]

\[ + E \omega^{MI}(r) \times \bar{I}^{MI}(r) \cdot E \omega^{MI}(r) \right] dr + \int_0^{BldFlexL} \left[ r^{QS1}(r) r^{QS1}(0) \right] \times \]

\[ F^{SI}_{AeroB1}(r) - \mu^{BI}(r) \left\{ g z_2 + \sum_{i=1}^{14} E v_i^{SI}(r) \ddot{q}_i \right\} \]

\[ + E v_i^{SI}(r) \ddot{q}_{Teet} + \sum_{i=16}^{19} \frac{d}{dt} (E v_i^{SI}(r)) \hat{q}_i \] +

\[ \frac{d}{dt} (E v_i^{SI}(r)) \hat{q}_{Teet} \right\} \] dr + \left[ r^{QS1}(BldFlexL) - r^{QS1}(0) \right] \times \]

\[ F^{SI}_{TipDragB1} - m^{BItip} \left\{ g z_2 + \sum_{i=1}^{14} E v_i^{SI}(BldFlexL) \ddot{q}_i \right\} + \ldots \]
Continuing the previous equation:

\[
M^H_{BI}(0) = \ldots + \\
\left[ \sum_{i=16}^{19} E_{\nu_i}^S(BldFlexL) \dot{q}_i \right] + E_{v_{Teet}}^S(BldFlexL) \ddot{q}_{Teet} + \\
\left[ \sum_{i=4}^{14} \frac{d}{dt} (E_{\nu_i}^S(BldFlexL)) \dot{q}_i \right] + \left[ \sum_{i=16}^{19} \frac{d}{dt} (E_{\nu_i}^S(BldFlexL)) \dot{q}_i \right] + \\
\frac{d}{dt} (E_{v_{Teet}}^S(r)) \dot{q}_{Teet} \right\} + \left[ \sum_{i=4}^{14} E_{\omega_i^M}^S(BldFlexL) \dot{q}_i \right] + E_{\omega_{Teet}^M}^S(BldFlexL) \ddot{q}_{Teet} + \\
\left[ \sum_{i=7}^{14} \frac{d}{dt} (E_{\omega_i^M}^S(BldFlexL)) \dot{q}_i \right] + \left[ \sum_{i=16}^{19} \frac{d}{dt} (E_{\omega_i^M}^S(BldFlexL)) \dot{q}_i \right] + \\
\frac{d}{dt} (E_{\omega_{Teet}^M}^S(BldFlexL)) \dot{q}_{Teet} \right\} - E_{\omega^M}^S(BldFlexL) \times \bar{I}_{B1Tip}^p.
\]

Breaking into components as in Eq. 3.37

\[
F^S_{Blr}(0) = - \int_0^{BldFlexL} \mu^B_1(r)E_{\nu^S_1}(r)dr - m^B_{1Tip} E_{v^S_1}(BldFlexL)
\]

\[ (r = 1, 2, \ldots, 14; 16, 17, 18; Teet) \]

and:
\[ F_{Bli}(0) = \]
\[ \int_{0}^{BldFlexL} \left\{ F_{AeroBli}^{SI}(r) - \mu^{Bli}(r) \left( g z_2 + \sum_{i=4}^{14} \frac{d}{dt} (E v_{i}^{SI}(r)) \dot{q}_i \right) + \right. \]
\[ \left. \left[ \sum_{i=16}^{19} \frac{d}{dt} \left( E v_{i}^{SI}(BldFlexL) \right) \dot{q}_i \right] + E v_{Teet}^{SI}(r) \ddot{q}_{Teet} \right\} dr - \]
\[ m^{Bli} \left( g z_2 + \sum_{i=4}^{14} \frac{d}{dt} \left( E v_{i}^{SI}(BldFlexL) \right) \dot{q}_i \right) + \]
\[ \left[ \sum_{i=16}^{19} \frac{d}{dt} \left( E v_{i}^{SI}(BldFlexL) \right) \dot{q}_i \right] + \frac{d}{dt} \left( E v_{Teet}^{SI}(BldFlexL) \right) \dot{q}_{Teet} \right\} + \]
\[ F_{TipDragBli}^{SI} \]

with the moment as in Eq. 3.38:

\[ M_{Bli}^{H}(0) = \]
\[ - \int_{0}^{BldFlexL} \left\{ -\vec{I}^{MI}(r) \cdot E \omega_{r}^{MI}(r) + \left[ r^{QS1}(r) r^{QS1}(0) \right] \times \right. \]
\[ \left[ \mu^{Bli}(r) E v_{r}^{SI}(r) \right] \right\} dr - m^{Bli} \left[ r^{QS1}(BldFlexL) - r^{QS1}(0) \right] \times \]
\[ E v_{r}^{SI}(BldFlexL) - \vec{I}^{Bli} \cdot E \omega_{r}^{MI}(BldFlexL) \]

\((r = 1, 2, \ldots, 14; 16, 17, 18, 19; Teet)\)
\[ M_{Bl1}^H(0) = \]
\[
\int_{BldFlexL}^B \left\{ -\overline{I}^{MI}(r) \cdot \left\{ \sum_{i=7}^{14} \frac{d}{dt} \left( E\omega_i^{MI}(r) \right) \dot{q}_i \right\} + \sum_{i=16}^{19} \frac{d}{dt} \left( E\omega_i^{MI}(r) \right) \dot{q}_i \right\} + \frac{d}{dt} \left( E\omega_{Teet}^{MI}(r) \right) \dot{q}_{Teet} \right\} - E\omega_i^{MI}(r) \times \overline{I}^{MI}(r) \cdot E\omega_i^{MI}(r) + [r^{QSI}(r) r^{QSI}(0)] \times \]
\[
\left\{ F_{AeroBl}(r) - \mu^{Bl}(r) \right\} \left\{ g z_2 + \sum_{i=4}^{14} \frac{d}{dt} \left( E v_i^{SI}(r) \right) \dot{q}_i \right\} + \sum_{i=16}^{19} \frac{d}{dt} \left( E v_i^{SI}(r) \right) \dot{q}_i \right\} \right\} dr + \]
\[
[r^{QSI}(BldFlexL) - r^{QSI}(0)] \times \left\{ F_{TipDragBl}^{SI} - \right\}
\[
\left\{ g z_2 + \sum_{i=4}^{14} \frac{d}{dt} \left( E v_i^{SI}(BldFlexL) \right) \dot{q}_i \right\} + \sum_{i=16}^{19} \frac{d}{dt} \left( E v_i^{SI}(BldFlexL) \right) \dot{q}_i \right\} \right\} + \]
\[
-\overline{I}^{BlTip} \cdot \left\{ \sum_{i=7}^{14} \frac{d}{dt} \left( E\omega_i^{MI}(BldFlexL) \right) \dot{q}_i \right\} + \sum_{i=16}^{19} \frac{d}{dt} \left( E\omega_i^{MI}(BldFlexL) \right) \dot{q}_i \right\} + \]
\[
\frac{d}{dt} \left( E\omega_{Teet}^{MI}(BldFlexL) \right) \dot{q}_{Teet} \right\} - E\omega^{MI}(BldFlexL) \times \overline{I}^{BlTip} \cdot \]
\[
E\omega^{MI}(BldFlexL) \]
3.5.2 Blade Gage Moment Outputs

The blade gage moments are computed in the subroutine CalcOuts. The equation for the moment at the gage location $R^{Span,j}$ is:

$$M_{B1}^{M1}(R^{Span,j}) =$$

$$\int_{R^{Span,j}}^{BldFlexL} \left[ M_{AeroB1}^{M1}(r) + \bar{I}^{M1}(r) \cdot E \alpha^{M1}(r) - E \omega^{M1}(r) \times \bar{I}^{M1}(r) \cdot E \omega^{M1}(r) \right] dr +$$

$$\int_{R^{Span,j}}^{BldFlexL} \left[ r^{QSI}(r) - r^{QSI}(R^{Span,j}) \right] \times \left[ F_{AeroB1}^{SI}(r) - \mu^{B1}(r) g z_2 - \mu^{B1}(r) E a^{SI}(r) \right] dr +$$

$$E \omega^{M1}(BldFlexL) \times \bar{I}^{B1}_{Tip} \times E \omega^{M1}(BldFlexL) +$$

$$\left[ r^{QSI}(BldFlexL) - r^{QSI}(R^{Span,j}) \right] \times \left\{ F_{TipDragB1}^{SI} - m^{B1}_{Tip} [g z_2 + E a^{SI}(BldFlexL)] \right\}$$

At the time of this writing, the author had not modified the blade gage moment outputs in CurveFAST.

3.6 Verification with FAST and Adams

For the initial CurveFAST development, the author used a straight blade with no taper with uniform inflow to ensure that aerodynamics and load were matching between CurveFAST, FAST and ADAMS. For the swept-blade verification, the author used the same “STAR7d” model that was used in the STAR program development [10]. The blade properties were proprietary and could not be included in this document. For the verification, the author compared 10-minute turbulent simulation runs at average speeds of ranging from 8-13 m/s. The author generated the wind files with NREL’s TurbSim program [18]; the file parameters are below in Table 3.3.
Table 3.3: Turbulent wind file parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random Number Generator</td>
<td>RANLUX</td>
</tr>
<tr>
<td>Vertical grid point dimension</td>
<td>13 m</td>
</tr>
<tr>
<td>Horizontal grid point dimension</td>
<td>13 m</td>
</tr>
<tr>
<td>Time step</td>
<td>0.05 s</td>
</tr>
<tr>
<td>Time series length</td>
<td>660 s</td>
</tr>
<tr>
<td>Usable length of time series</td>
<td>630 s</td>
</tr>
<tr>
<td>Hub height</td>
<td>61 m</td>
</tr>
<tr>
<td>Grid Height</td>
<td>65 m</td>
</tr>
<tr>
<td>Grid Width</td>
<td>65 m</td>
</tr>
<tr>
<td>Vertical mean flow angle</td>
<td>8°</td>
</tr>
<tr>
<td>Turbulence spectrum</td>
<td>Kaimal</td>
</tr>
<tr>
<td>IEC turbulence class</td>
<td>A</td>
</tr>
</tbody>
</table>

The two programs used the same turbulent wind file for the associated average wind speed. The turbine models, including the blade pitch and controller model, were identical. The simulations did not include flexible tower and drivetrain degrees of freedom. The comparisons were with blade tip deflection, generator power, tip twist, blade flap bending moment, and blade edge bending moment.

The author built the Adams models using the FAST2ADAMS preprocessor that he modified during the STAR program. These modifications changed the alignment of the structural axis from parallel to the pitch (or reference axis) to parallel to the elastic axis of the curved blade. These changes were necessary to obtain the correct pitch bearing moments for blades with sweep at the pitch bearing. Also, the author found during the current verification that these changes were necessary to obtain correct values for static deflection under gravity loads. The author documented these changes in unpublished report (titled “Changes to FAST2ADAMS for Blades with Precurve/Presweep”, dated April 10, 2006) submitted to Jason Jonkman at NREL.

The maximum percentage difference for several parameters for the CurveFAST and Adams STAR7d comparison are shown below in Table 3.4.
Table 3.4: Maximum percentage differences for the Adams/CurveFAST verification

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Statistic</th>
<th>Maximum Percentage Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade 1 tip twist</td>
<td>Average</td>
<td>-13.3%</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>-145%</td>
</tr>
<tr>
<td></td>
<td>Minimum</td>
<td>21.2%</td>
</tr>
<tr>
<td>Blade 1 tip deflection</td>
<td>Average</td>
<td>6.7%</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>12.6%</td>
</tr>
<tr>
<td>Generator power</td>
<td>Average</td>
<td>4.1%</td>
</tr>
<tr>
<td>Blade 1 flap bending moment</td>
<td>Average</td>
<td>3.0%</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>9.1%</td>
</tr>
<tr>
<td>Blade 1 edge bending moment</td>
<td>Maximum</td>
<td>2.8%</td>
</tr>
<tr>
<td></td>
<td>Minimum</td>
<td>3.7%</td>
</tr>
</tbody>
</table>

Results for the parameters in Table 3.4 plotted versus wind speed are below in Figures 3.5 to Figure 3.9. The parameters were normalized by the author by the maximum value in the plot.
The figure above shows consistently more tip twist using the Adams methodology.
Figure 3.6: Blade tip deflection verification

The figure above shows consistently more blade tip deflection for the present (Curve-FAST) methodology, with a maximum deflection occurring near the 11 m/s rated wind speed.
The generator power agreement is within 5%, but with lower power for the Adams methodology in Region 2 (wind speed below wind speed for rated power) of the power curve.
Figure 3.8: Flap bending moment verification

The agreement for flap bending is within 5% except for the maximum near the rated wind speed of 11 m/s.
Figure 3.9: Edge bending moment verification

The agreement for edge bending is within 5% across the range.
The author performed additional verification with the extreme coherent gust with direction change, which most likely results in the maximum blade tip-deflection, as shown in Larwood and Zuteck [10].

Table 3.5: Extreme Load Verification, Normalized Maximum Values

<table>
<thead>
<tr>
<th>Channel</th>
<th>ADAMS</th>
<th>CurveFAST</th>
<th>Δ%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade deflection</td>
<td>0.957</td>
<td>1.000</td>
<td>4.3%</td>
</tr>
<tr>
<td>Flap bending</td>
<td>0.964</td>
<td>1.000</td>
<td>3.6%</td>
</tr>
<tr>
<td>Tip twist</td>
<td>1.000</td>
<td>0.811</td>
<td>18.9%</td>
</tr>
</tbody>
</table>

Because of the apparent lower twist and loads from the Adams simulation, the author ran a linear analysis of the Adams model to compare the modes shapes with the CurveFEM mode shapes at zero rotational speed. The out-of-plane and torsional displacements for the first flap mode are below in Figure 3.10 and Figure 3.11 respectively. The displacements are normalized to the maximum out-of-plane displacement.

These plots show close agreement for the out-of-plane displacement, but Adams shows more twist motion in the first flap mode. The author obtained similar results for the second flap mode.
Figure 3.10: Out-of-plane deflection for STAR7d first flap bending mode
Figure 3.11: Torsional deflection for STAR7d first flap bending mode
3.7 Validation with Field Test Data

The STAR program field test occurred in the first and second quarters of 2008. The test machine was a Zond Z-48 with a 750 kW rating, located in Tehachapi, California. The STAR test rotor was 54-meters in diameter, down two meters from the original design. The original STAR design would fit a Zond Z-50 machine. Due to budget constraints, the test program did not include model validation; therefore, the test team installed minimal instrumentation. Along with verifying safe operation with the STAR rotor, the test goals were to obtain root bending loads and output power for comparison to tests of baseline Z-48 turbines.

The author obtained binned-average field test data for model comparisons. The data included generator power, blade root flap- and edge-bending of a single blade, and blade pitch position of a single blade. The data ranged from average wind speeds of 2 m/s to 16 m/s. The wind speed was measured directly from the nacelle anemometer, which most likely did not represent the ambient wind speed. It most likely was a reduced-speed measurement of ambient wind due to the presence of the turbine wake.

The author performed validation runs using the density at the site altitude. The runs were five 10-minute turbulent wind simulations at each wind speed, using the Kaimal spectrum and matching the average turbulent intensity at each wind speed bin from the test data. Using the test turbulence intensity was necessary because it was much lower than IEC A or B model turbulence. The author had no detailed information regarding the generator or pitch control, so the same control model was used from the STAR7d development. For the same reasons the author disabled the flexible tower and drivetrain degrees of freedom.

The comparison plots for generator power, blade pitch, root edge bending, and root flap bending are below respectively in Figure 3.12 to Figure 3.15. The author normalized the data to an arbitrary value slightly above the maximum value of the plot.
Figure 3.12: Generator power validation

This plot shows the power prediction based on CurveFAST to be lower at first then exceeding the test results before rated wind speed.
Figure 3.13: Blade pitch validation

Figure 3.13 shows the CurveFAST controller pitching at a higher wind speed and with a higher slope than the turbine controller.
Figure 3.14: Edge bending validation

Figure 3.14 shows CurveFAST predicting edge bending loads at the blade root below measured loads with the predicted peak loads falling 14.1% below the peak measured loads.
Figure 3.15 shows CurveFAST moment lower just below rated then overshooting at rated. The author performed runs at higher fine pitch (the pitch position below rated power) and this peak was lowered to the test data. The peak flap load is 2.3% from the peak measured load.

3.8 Conclusions

3.8.1 Verification with Adams

The author concluded that the agreement with Adams in the verification results was acceptable. The author based his conclusion on several points:

1. The agreement was mostly within 5% for parameters that are normal test measure-
ments, such as flap and edge bending, and generator power [52]. Blade deflection and tip twist are not normal test measurements and would be difficult for researchers to obtain. The one exception is the maximum difference in maximum flap bending (see Table 3.4); however, this difference is at the wind speed where the turbine reaches rated power. The loads can fluctuate rapidly in this region due to the controller action. Better agreement may have been obtained by the author if more runs were conducted at this wind speed.

2. The large disagreement in blade tip twist is due to the low absolute values of this parameter.

3. The results do show that Adams produces more twist and lower loads; however, it is not clear whether this is due to non-linearities captured by Adams, inclusion of higher-order modes, or the lumped-parameter model. Based on Figure 3.11, the author concluded that the difference may not be due to non-linearities or higher modes, because for a linearized Adams model the twist degree of freedom shows higher amplitude than the finite-element model for the first flap bending mode.

4. To place this verification in perspective, Buhl and Wright reported on verification between FAST and Adams for a straight blade turbine [53] at uniform wind speed. The results in this paper show some difference in the flap bending, and the difference in rotor torque was 6%.

3.8.2 Validation with Field Test Data

The author concluded that the test results could not either validate CurveFAST and CurveFEM. There were several reasons contributing to the discrepancy between the test and modeling results. First, the upstream wind speed was not measured. Therefore, the test data curves would shift for this correction. Secondly, the torque-speed curve was not known. This would shift the model flap, edge, and power curves. Thirdly, the model pitch
controller is not matching the behavior of the test pitch controller. Even with a shift in the actually wind speed, the test turbine is most likely pitching at a lower wind speed compared to the model turbine. Capturing this behavior would lower the peak mean flap for the model at rated wind speed shown in Figure 3.15. However, the model is matching the peak flap load. Finally, the model is missing the peak edge and flap loads. The test edge-loads may be due to coalescing of the first edge mode with a multiple of the rotor rotational speed.

### 3.9 Recommendations

**Validation Effort** The STAR test program has tentatively shown that a baseline rotor can be expanded with a swept design. The program also proved that the swept rotor can be constructed with current technology. The author highly recommends a more thorough validation effort for the swept-blade concept, which would help validate both the CurveFAST and Adams modeling. An ideal test program would be one were there is a baseline turbine with validated models, and then the rotor is changed out with a swept design.

**Updating FAST** The author recommends that NREL consider incorporating into a future version of FAST the new features available in CurveFAST. This would allow more users to test the model and incorporate the swept-blade concept into their designs.

**Model Upgrades** Another recommendation is to enhance the model by incorporating blade center-of-gravity and elastic-axis offsets into both CurveFAST and CurveFEM.

**Further Adams Verification** One further verification effort could be to develop the finite-element model with a lumped-mass matrix (as opposed to a consistent mass matrix, see Rao [28]), and use the modes for comparison to Adams.
Chapter 4

Design Studies

4.1 Introduction

With the modified and validated FAST, the author studied the feasibility of the swept-blade concept for key design parameters, scaling to larger rotors, and flutter stability boundaries.

The sweep of the STAR blade design is controlled by an equation that incorporates the start of the blade sweep along the rotor axis, the exponent of the sweep curve, and the amount of sweep at the tip. The section torsional stiffness is also critical for the design. These parameters are constrained by manufacturing and transportation limitations. For example, manufacturing is affected by the shape of the sweep curve because the construction materials, primarily woven fiberglass sheets, have limits to their curvature. Tip sweep is affected by transportation requirements, and for the LWST program the tip sweep was limited to the envelope of maximum chord. The STAR program team conducted a cursory design study by changing the sweep parameters individually for an individual load case at rated wind speed. For the concept feasibility studies, the author performed a more thorough investigation to study the impact on the sweep parameters. The study determined the impact on both the loads and annual energy production.
With an understanding of the key design parameters for the swept blade concept, the author studied the feasibility of scaling to larger rotors. The first step was scaling the rotor to a 1.5 MW turbine, which is the current workhorse for General Electric. It should be noted that GE was working on a swept blade design for this machine. With the WindPACT studies, NREL has available design details of a 1.5 MW machine in the public domain. The author used this model as a baseline for incorporating the swept-blade concept. The design evaluation used the same key load cases used for the 750 kW machine.

From this point, the author evaluated a turbine of 3.0 MW rating. Currently, the largest land-based production unit is the Vestas V-90 at 3 MW. These machines were recently installed in Solano County. The author used the WindPACT 3.0 MW turbine design for this study.

The design goal of the feasibility studies was to determine if by incorporating blade sweep the Annual Energy Production (AEP) can be increased by 5% over the baseline design.

For all three designs, the flutter stability was studied using the method by Lobitz [27]. Running the simulation at rated wind speed, the rotor speed is increased until the instability is encountered, which is manifested in oscillations in tip twist. This method compared favorably to a more complex Theodorsen frequency domain technique.

### 4.2 Baseline Studies

#### 4.2.1 Method

The baseline model was the STAR7d from the original STAR program. The STAR team designed the blade for the Zond Z-50 turbine [10]. The model data is proprietary so the author cannot disclose design details; therefore; the results are normalized. Table 4.1 gives a listing of the parametric studies.

For the torsional stiffness study, a constant value (either 0.8 or 1.2) multiplied the tor-
Table 4.1: Parametric Studies

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Percentage from Baseline</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsional Stiffness</td>
<td>+20%</td>
</tr>
<tr>
<td>Torsional Stiffness</td>
<td>-20%</td>
</tr>
<tr>
<td>Sweep Exponent</td>
<td>+20%</td>
</tr>
<tr>
<td>Sweep Exponent</td>
<td>-20%</td>
</tr>
<tr>
<td>Tip Sweep</td>
<td>+15%</td>
</tr>
<tr>
<td>Tip Sweep</td>
<td>-15%</td>
</tr>
</tbody>
</table>

Torsional stiffness at each blade element. For the sweep exponent study, the equation for blade sweep was from Eq. 3.27, where a constant value (either 0.8 or 1.2) multiplied the sweep exponent. Similarly, a constant value (either 0.85 or 1.15) multiplied the tip sweep. The author limited the upper end of the sweep exponent and tip sweep so the sweep angle did not exceed 20°.

Each model run was a 10-minute turbulent simulation, which had a nominal 10-minute average wind speed from 3 m/s to 25 m/s in 2 m/s steps. The NREL program TurbSim [18] generated the wind files with the Kaimal spectrum for the IECA normal turbulence model. Each wind speed step consisted of five simulations with the random seed equal to the system clock.

The author compared the NTM results by calculating the damage equivalent load (DEL), the annual energy production (AEP), and the maximum blade deflection.

The damage equivalent load is the load (or moment in this case) for a given number of cycles (one million in this case) that will produce the same damage for the lifetime load spectra. Freebury and Musial [54] and Sutherland [55] discuss this method. The calculation of the DEL involves the use of Miner’s rule, which is:

\[
\frac{n_1}{N_1} + \frac{n_2}{N_2} + \cdots + \frac{n_n}{N_n} = \text{Damage Fraction},
\]

where \( n \) is the number of cycles for a particular stress level, and \( N \) is the number of allowable cycles for a particular stress level. The damage fraction for failure is normally unity (1.0). For the equivalent damage of one million cycles, Eq. 4.1 becomes:
The number of allowable cycles, $N$, is from the Wöhler (or $S$-$N$) curve:

$$N = \left( \frac{\sigma_a}{\sigma_u} \right)^{-m}, \quad (4.3)$$

where $\sigma_a$ is the applied stress, $\sigma_u$ is the ultimate stress, and $m$ is the inverse of the Wöhler (or $S$-$N$) curve. The DEL analysis assumes a linear relation between stress and load (or moment in this case). Equation 4.4 then becomes:

$$N = \left( \frac{M_a}{M_u} \right)^{-m}, \quad (4.4)$$

where $M_a$ is the applied moment and $M_u$ is the ultimate moment. Equation 4.4 is substituted into Eq. 4.2 to obtain:

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \cdots + \frac{n_n}{N_n} = \frac{1 \times 10^6}{N_{eq}} \quad (4.5)$$

where $M_{eq}$ is the damage equivalent moment for one million cycles. Solving for $M_{eq}$:

$$M_{eq} = \left( \frac{\sum (M_{ai})^{m} n_i}{1 \times 10^6} \right)^{1/m}, \quad (4.6)$$

The rainflow cycle counting algorithm in NREL’s Crunch program [56] determines the number of cycles per simulation run for thirty moment levels. The author wrote an Excel program to use the rainflow counts to compute the DEL. The program adds up the cycles for the five 10-minute runs at each average wind speed step. It then multiplies these counts by $6/5$ and by the annual number of hours for the wind speed assuming a Rayleigh distribution. The Rayleigh probability distribution is from:
\[ F(V) = 1 - \exp \left( -\frac{\pi}{4} \left( \frac{V}{V_{ave}} \right)^2 \right), \quad (4.7) \]

where \( F(V) \) is the frequency that the velocity \( V \) or lower will occur, and \( V_{ave} \) is the average wind speed of the distribution. The author used an annual average wind speed of 8.5 m/s, which corresponds to IEC class II. The number of hours at wind speed \( V_i \) is from:

\[
\text{Hours at wind speed } V_i = 8760(F(V_i) - F(V_{i-1})), \quad (4.8)
\]

with the calculation initiated at:

\[
V_{i-1} = V_i - 1 \text{ m/s} \quad (4.9)
\]

The calculation for the annual energy production (AEP) averages the generator power of the five simulations at a particular average wind speed, and then multiplies by the number of hours at that wind speed using Eq. 4.8. The total sum of these kilowatt-hours is the AEP.

The peak deflection is simply the maximum deflection from the model runs.

To determine the distribution in AEP and DEL for the baseline model, the author ran 5 separate 60-case runs. The standard deviation of AEP was 0.4%. The standard deviation of flap and edge DEL (for Wöhler slope of 10) were 1.4% and 0.15% respectively.

For the flutter study, the author followed the work of Lobitz [57] who had found that flutter could be initiated using Adams. Lobitz had determined a flutter speed for the STAR rotor using the classical approach with a finite-element code and the Theodorsen function; however, these results are considered proprietary. The author conducted the study for this project using the STAR7d model in calm conditions. The rotor speed was varied from rated to a point above rated using an induction-generator model with the rated speed set above the nominal rated-speed. As in the Lobitz study, the simulation did not include the airfoil moment coefficient, had blade damping at 0.05\% critical, and used the Beddoes-Leishman dynamic stall model, described the AeroDyn theory manual [48].
4.2.2 Results

Figures 4.1, 4.2 and 4.3 show the variation in DEL, AEP, and maximum blade 1 deflection respectively for the parametric studies in Table 4.1.

![Graph showing variations in DEL, AEP, and maximum blade 1 deflection for different parametric studies.](image)

Figure 4.1: Parametric study of damage equivalent loads

These results show an increase in flap DEL with torsional stiffness, and a decrease in DEL with tip sweep. The sweep exponent has minimal effect on the flap DEL.
These results show less than 1% variation in AEP for the range of the parameters studied. The AEP increases with torsional stiffness and sweep exponent, but decreases with tip sweep.

These results show an increase in blade deflection with torsional stiffness. The results for tip sweep and sweep exponent are not conclusive.

For the flutter study, the CurveFAST model showed a slow exponential rise in blade tip twist from the start of the simulation. The same model run in Adams did not exhibit this behavior. The CurveFAST model without aerodynamic loads also did not exhibit this behavior. With aerodynamic loads, the CurveFAST model exhibited stable behavior with the 0.2% blade damping, which is still below typical values for blade structural damping [6]. The models did not indicate the flutter speed indicated by Lobitz.

Figure 4.2: Parametric study of annual energy production
Figure 4.3: Parametric study of blade tip deflection
4.2.3 Conclusions and Discussion

The parametric study results show the importance of low torsional stiffness to reduce fatigue loads (DEL) and blade deflection. Both Liebst [9] and Zuteck [8] discuss the importance of lowering the torsional stiffness to achieve load relief.

Higher tip sweep reduces fatigue loads (Fig. 4.1); however, the results do not show whether or not blade sweep decreases tip deflection. For future study, the IEC extreme load cases in addition to the turbulent operating cases should be run to study changes in maximum blade deflection with tip sweep.

Modifying the sweep curve exponent has less impact on loads. Note that the sweep remains in the outboard blade portion, which Zuteck [8] notes as important. He shows that a circular curve extending inward to the root is less effective than more outboard curvature.

For all of studies, the change in annual energy production is less than 1% (Fig. 4.2). This value is less than the expected accuracy for power performance tests. Therefore, changing the parameters within the range of this study would probably have no noticeable effect on the annual energy production.

The flutter study was not successful in showing a flutter speed that was calculated by a classical approach. Researchers should investigate the analytical approach to flutter by Lobitz [57], who used NASTRAN to calculate the flutter speed; another approach would be to adapt the author’s CurveFEM code for these calculations.

4.3 Scaling to Larger rotors

4.3.1 Method

For the scaling studies, the author used the baseline WindPACT models [58] of 1.5 MW and 3.0 MW size. The 1.5 MW is included in the FAST code distribution [16]. Craig Hansen, who was involved in the WindPACT studies, provided the 3.0 MW model data.
The airfoil tables for these models do not include the sectional moment coefficient, so the author disabled the first torsion mode degree-of-freedom for the study. The author ran the baseline models in the same manner as the parametric studies to obtain annual energy production and damage equivalent loads. Table 4.2 shows the WindPACT model parameters.

<table>
<thead>
<tr>
<th>Model</th>
<th>WP1500</th>
<th>WP3000</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rated power, kW</strong></td>
<td>1500</td>
<td>3000</td>
</tr>
<tr>
<td><strong>Radius, m</strong></td>
<td>35.5</td>
<td>49.5</td>
</tr>
<tr>
<td><strong>Maximum chord, m</strong></td>
<td>2.8</td>
<td>3.96</td>
</tr>
<tr>
<td><strong>Rated generator speed, rpm</strong></td>
<td>1800</td>
<td>1800</td>
</tr>
<tr>
<td><strong>Rated rotor speed, rpm</strong></td>
<td>20.463</td>
<td>14.469</td>
</tr>
<tr>
<td><strong>Hub radius, m</strong></td>
<td>1.65</td>
<td>2.325</td>
</tr>
<tr>
<td><strong>Hub height, m</strong></td>
<td>84</td>
<td>119</td>
</tr>
<tr>
<td><strong>Gearbox efficiency, %</strong></td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td><strong>Generator efficiency, %</strong></td>
<td>95</td>
<td>95</td>
</tr>
<tr>
<td><strong>Gearbox ratio</strong></td>
<td>87.965</td>
<td>124.4</td>
</tr>
</tbody>
</table>
Table 4.3 shows the blade properties for the WindPACT 1.5 MW turbine.

Table 4.3: WP1500 blade properties

<table>
<thead>
<tr>
<th>Station</th>
<th>Blade Fraction</th>
<th>Twist (deg)</th>
<th>Chord (m)</th>
<th>Pitch Axis Ratio</th>
<th>Aero Center</th>
<th>Mass (kg/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000</td>
<td>11.1</td>
<td>1.925</td>
<td>0.500</td>
<td>0.50</td>
<td>1447.61</td>
</tr>
<tr>
<td>2</td>
<td>0.0211</td>
<td>11.1</td>
<td>1.890</td>
<td>0.500</td>
<td>0.50</td>
<td>173.89</td>
</tr>
<tr>
<td>3</td>
<td>0.2105</td>
<td>11.1</td>
<td>2.800</td>
<td>0.340</td>
<td>0.25</td>
<td>204.04</td>
</tr>
<tr>
<td>4</td>
<td>0.4737</td>
<td>3.1</td>
<td>2.147</td>
<td>0.310</td>
<td>0.25</td>
<td>157.61</td>
</tr>
<tr>
<td>5</td>
<td>0.7368</td>
<td>0.6</td>
<td>1.494</td>
<td>0.280</td>
<td>0.25</td>
<td>72.66</td>
</tr>
<tr>
<td>6</td>
<td>1.0000</td>
<td>0</td>
<td>0.906</td>
<td>0.250</td>
<td>0.25</td>
<td>11.35</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Station</th>
<th>Flatwise Stiffness (N · m²)</th>
<th>Edgewise Stiffness (N · m²)</th>
<th>Torsional Stiffness (N · m²)</th>
<th>AE Product (N)</th>
<th>Airfoil Filename</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.6815E+09</td>
<td>7.6815E+09</td>
<td>2.6552E+09</td>
<td>1.7153E+10</td>
<td>cylinder</td>
</tr>
<tr>
<td>2</td>
<td>1.1281E+09</td>
<td>1.1281E+09</td>
<td>3.9418E+08</td>
<td>2.5464E+09</td>
<td>cylinder</td>
</tr>
<tr>
<td>3</td>
<td>3.0477E+08</td>
<td>6.4782E+08</td>
<td>1.9215E+07</td>
<td>2.7043E+09</td>
<td>s818_2703.dat</td>
</tr>
<tr>
<td>4</td>
<td>8.5919E+07</td>
<td>2.7108E+08</td>
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<td>2.0742E+09</td>
<td>s818_2703.dat</td>
</tr>
<tr>
<td>5</td>
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<td>7.0329E+07</td>
<td>1.6868E+06</td>
<td>9.2581E+08</td>
<td>s825_2103.dat</td>
</tr>
<tr>
<td>6</td>
<td>2.3129E+05</td>
<td>7.8741E+06</td>
<td>1.7943E+05</td>
<td>1.1847E+08</td>
<td>s826_1603.dat</td>
</tr>
</tbody>
</table>
Table 4.4 shows the blade properties for the WindPACT 3.0 MW turbine.

### Table 4.4: WP3000 blade properties

<table>
<thead>
<tr>
<th>Station</th>
<th>Blade Fraction</th>
<th>Twist (deg)</th>
<th>Chord (m)</th>
<th>Pitch Axis Ratio</th>
<th>Aero Center</th>
<th>Mass (kg/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000</td>
<td>11.1</td>
<td>2.673</td>
<td>0.500</td>
<td>0.50</td>
<td>2514.27</td>
</tr>
<tr>
<td>2</td>
<td>0.0211</td>
<td>11.1</td>
<td>2.673</td>
<td>0.500</td>
<td>0.50</td>
<td>342.34</td>
</tr>
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<td>3</td>
<td>0.2105</td>
<td>11.1</td>
<td>3.960</td>
<td>0.340</td>
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<td>373.58</td>
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<td>0.4737</td>
<td>3.1</td>
<td>3.036</td>
<td>0.310</td>
<td>0.25</td>
<td>302.91</td>
</tr>
<tr>
<td>5</td>
<td>0.7368</td>
<td>0.6</td>
<td>2.113</td>
<td>0.280</td>
<td>0.25</td>
<td>136.39</td>
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<tr>
<td>6</td>
<td>1.0000</td>
<td>0</td>
<td>1.281</td>
<td>0.250</td>
<td>0.25</td>
<td>16.68</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Station</th>
<th>Flatwise Stiffness (N · m²)</th>
<th>Edgewise Stiffness (N · m²)</th>
<th>Torsional Stiffness (N · m²)</th>
<th>AE Product (N)</th>
<th>Airfoil Filename</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.5916E+10</td>
<td>2.5916E+10</td>
<td>8.9665E+09</td>
<td>2.8944E+10</td>
<td>cylinder</td>
</tr>
<tr>
<td>2</td>
<td>4.5189E+09</td>
<td>4.5189E+09</td>
<td>1.5791E+09</td>
<td>5.0974E+09</td>
<td>cylinder</td>
</tr>
<tr>
<td>3</td>
<td>1.3320E+09</td>
<td>3.4791E+09</td>
<td>6.2408E+07</td>
<td>4.9586E+09</td>
<td>s818_2703.dat</td>
</tr>
<tr>
<td>4</td>
<td>3.4479E+08</td>
<td>9.6503E+08</td>
<td>2.5625E+07</td>
<td>4.0027E+09</td>
<td>s818_2703.dat</td>
</tr>
<tr>
<td>5</td>
<td>5.4736E+07</td>
<td>2.4072E+08</td>
<td>5.1116E+06</td>
<td>1.7456E+09</td>
<td>s825_2103.dat</td>
</tr>
<tr>
<td>6</td>
<td>6.9195E+05</td>
<td>2.3690E+07</td>
<td>5.4377E+05</td>
<td>1.7586E+08</td>
<td>s826_1603.dat</td>
</tr>
</tbody>
</table>
The author scaled the baseline turbines for swept blades by increasing the rotor swept area by 25% percent. This was an approximate method for obtaining an increase of 5% AEP. The new model’s maximum chord was the same as the baseline, which was at the 25% radial position. To optimize the new blade planform, the author used the Betz method outlined in Gasch and Twele [5]. For this method, the author scaled the WindPACT optimal tip-speed ratio (7) by the ratio of the new rotor radius to the baseline rotor radius. The new blade used the maximum chord of the baseline blade at 25% radial station, and the Betz-optimized chord at the 75% radial station. The chord distribution was linearly tapered from these two positions, with the tip chord not falling under 300 mm. The twist distribution was from the Betz-optimized twist distribution based on the maximum L/D value and corresponding angle of attack for the 75% radial position airfoil. The maximum twist was 11.1 degrees, which was the WindPACT value.

The new blade’s stiffness properties were the same as the baseline at the same blade station normalized by the blade length. This was approximately the same situation with the STAR7d and its baseline blade. The only exception was the torsional stiffness, which the author varied in the design process. The author adjusted the blade lineal density by a common factor so that the static moment was the same for the baseline and the extended blade. The STAR program was able to accomplish this matching of the blade static moments.

The author ran simulations of the new rotors with the WT_Perf performance code [59] to determine the optimal pitch, tip speed ratio, and power coefficient. The author then used these settings to determine the variable speed control parameters for CurveFAST. For the pitch control, the author used WindPACT pitch control model and ran simulations with stepped wind speeds above rated. The author adjusted the proportional gain as necessary to ensure stable behavior.

The author ran normal turbulence model simulations as for the baseline machines, with annual energy production and damage equivalent loads computed in post-processing. The author compared the results to the baseline and adjusted turbine design parameters in order
to obtain a 5% AEP increase with no increase in flap bending DELs.

### 4.3.2 Results

Table 4.5 lists the resulting designs, performance, and loads for the two scaled rotor models, Curve1500 and Curve3000.

<table>
<thead>
<tr>
<th>Model</th>
<th>Curve1500</th>
<th>Curve3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline WindPACT radius, m</td>
<td>35.5</td>
<td>49.5</td>
</tr>
<tr>
<td>Stretched radius, m</td>
<td>39.15</td>
<td>55.35</td>
</tr>
<tr>
<td>Maximum chord, m</td>
<td>2.8</td>
<td>3.96</td>
</tr>
<tr>
<td>Tip sweep, m</td>
<td>2.8</td>
<td>2.376</td>
</tr>
<tr>
<td>GJ modification</td>
<td>74% lower</td>
<td>same as baseline</td>
</tr>
<tr>
<td>Percent AEP over baseline</td>
<td>5.0%</td>
<td>5.1%</td>
</tr>
<tr>
<td>Percent Flap DEL under baseline (m = 10)</td>
<td>6.6%</td>
<td>10.2%</td>
</tr>
<tr>
<td>Percent Edge DEL over baseline (m = 10)</td>
<td>3.7%</td>
<td>1.7%</td>
</tr>
</tbody>
</table>

The table shows that the AEP has increased 5% or higher for the swept-blade designs, with a decrease in the flap-bending DEL. The larger turbine required less tip sweep and no reduction in torsional stiffness to achieve the design goal. Both designs show an increase in edge DEL.
Figure 4.4 below shows the flap DEL versus wind speed for the Curve1500 model and a fatigue-curve slope $m = 10$. The author has normalized the DEL at an arbitrary value slightly above the maximum value.

![Figure 4.4: Flap bending comparison for scaled design](image)

The results in the figure show for the swept rotor that the flap DEL is higher below rated wind speed but falls below the straight rotor above rated.
4.3.3 Conclusions and Recommendations

The new swept designs lower flap bending loads compared to the baseline straight blades and provide increased energy capture. The swept-blade concept does not seem to suffer at larger scales, with the required torsional flexibility and tip sweep being lower in comparison to smaller scale. However, the swept blade concept does not offer relief for the edge bending loads, which will become critical for larger turbines.

Given the tentative success of this project and the successful operation of the test turbine, the author recommends that this concept should be explored further for commercial application. The author recommends continuing the design studies to a more detailed effort which would include detailed section properties, and running of the full IEC extreme load cases to ensure that baseline loads are not exceeded.
Bibliography


Appendix A

Blade Finite Element Matrices

This appendix lists the equations for the elements of the blade finite element matrices. The elements are from the upper triangle. Elements from the upper triangle that are not listed assume the value zero. The matrices are symmetric or skew-symmetric as noted.

A.1 Mass Matrix

The element mass matrix $[M]$ is symmetric, with the elements:

\[ M_{1,1} = l/35(10\mu_0 + 3\mu_l) \]
\[ M_{1,5} = l^2/420(15\mu_0 + 7\mu_l) \]
\[ M_{1,7} = 9l/140(\mu_0 + \mu_l) \]
\[ M_{1,11} = -l^2/420(7\mu_0 + 6\mu_l) \]
\[ M_{2,2} = l/35(10\mu_0 + 3\mu_l) \]
\[ M_{2,4} = -l^2/420(15\mu_0 + 7\mu_l) \]
\[ M_{2,8} = 9l/140(\mu_0 + \mu_l) \]
\[ M_{2,10} = \frac{l^2}{420}(7\mu_0 + 6\mu_l) \]
\[ M_{3,3} = \frac{l}{12}(3\mu_0 + \mu_l) \]
\[ M_{3,9} = \frac{l}{12}(\mu_0 + \mu_l) \]
\[ M_{4,4} = \frac{l^3}{840}(5\mu_0 + 3\mu_l) \]
\[ M_{4,8} = -\frac{l^2}{420}(6\mu_0 + 7\mu_l) \]
\[ M_{4,10} = -\frac{l^3}{280}(\mu_0 + \mu_l) \]
\[ M_{5,5} = \frac{l^3}{840}(5\mu_0 + 3\mu_l) \]
\[ M_{5,7} = \frac{l^2}{420}(6\mu_0 + 7\mu_l) \]
\[ M_{5,11} = -\frac{l^3}{280}(\mu_0 + \mu_l) \]
\[ M_{6,6} = \frac{l}{12}(3J_0' + J_1') \]
\[ M_{6,12} = \frac{l}{12}(J_0' + J_1') \]
\[ M_{7,7} = \frac{l}{35}(3\mu_0 + 10\mu_l) \]
\[ M_{7,11} = -\frac{l^2}{420}(7\mu_0 + 15\mu_l) \]
\[ M_{8,8} = \frac{l}{35}(3\mu_0 + 10\mu_l) \]
\[ M_{8,10} = \frac{l^2}{420}(7\mu_0 + 15\mu_l) \]
\[ M_{9,9} = \frac{l}{4}(\mu_0/3 + \mu_l) \]
\[ M_{10,10} = \frac{l^3}{840}(3\mu_0 + 5\mu_l) \]
\[ M_{11,11} = \frac{l^3}{840}(3\mu_0 + 5\mu_l) \]
A.2 Elastic Stiffness Matrix

The element elastic stiffness matrix $[K_e]$ is symmetric, with the elements:

\[ K_{e1,1} = \frac{6}{l^3}(EI_{y0} + EI_{yl}) \]
\[ K_{e1,5} = \frac{2}{l^2}(2EI_{y0} + EI_{yl}) \]
\[ K_{e1,7} = -\frac{6}{l^3}(EI_{y0} + EI_{yl}) \]
\[ K_{e1,11} = \frac{2}{l^2}(EI_{y0} + 2EI_{yl}) \]
\[ K_{e2,2} = \frac{6}{l^3}(EI_{x0} + EI_{xt}) \]
\[ K_{e2,4} = -\frac{2}{l^2}(2EI_{x0} + EI_{xt}) \]
\[ K_{e2,8} = -\frac{6}{l^3}(EI_{x0} + EI_{xt}) \]
\[ K_{e2,10} = -\frac{2}{l^2}(EI_{x0} + 2EI_{xt}) \]
\[ K_{e3,3} = \frac{(EA_0 + EA_l)}{2l} \]
\[ K_{e3,9} = \frac{-(EA_0 + EA_l)}{2l} \]
\[ K_{e4,4} = \frac{1}{l}(3EI_{x0} + EI_{xt}) \]
\[ K_{e4,8} = \frac{2}{l^2}(2EI_{x0} + EI_{xt}) \]
\[ K_{e4,10} = \frac{1}{l}(EI_{x0} + EI_{xt}) \]
\[ K_{e5,5} = \frac{1}{l}(3EI_{y0} + EI_{yl}) \]
\[ K_{e5,7} = -\frac{2}{l^2}(2EI_{y0} + EI_{yl}) \]
\[ K_{e5,11} = \frac{1}{l}(EI_{y0} + EI_{yl}) \]
\[ K_{e6,6} = (G_{J0} + G_{J1})/2l \]

\[ K_{e6,12} = -(G_{J0} + G_{J1})/2l \]

\[ K_{e7,7} = 6/l^3(EI_{y0} + EI_{yl}) \]

\[ K_{e7,11} = -2/l^2(EI_{y0} + 2EI_{yl}) \]

\[ K_{e8,8} = 6/l^3(EI_{x0} + EI_{xl}) \]

\[ K_{e8,10} = 2/l^2(EI_{x0} + 2EI_{xl}) \]

\[ K_{e9,9} = (EA_{0} + EA_{1})/2l \]

\[ K_{e10,10} = 1/l(EI_{x0} + 3EI_{xl}) \]

\[ K_{e11,11} = 1/l(EI_{y0} + 3EI_{yl}) \]

\[ K_{e12,12} = (G_{J0} + G_{J1})/2l \]

### A.3 Gyroscopic Matrix

The element gyroscopic matrix \([G]\) is skew-symmetric, with the elements:

\[ G_{1,2} = (156b_{1})\mu_{0}l\Omega/420 + (72b_{1})(\mu_{t} - \mu_{0})l\Omega/840 \]

\[ G_{1,3} = (147b_{2})\mu_{0}l\Omega/420 + (70b_{2})(\mu_{t} - \mu_{0})l\Omega/840 \]

\[ G_{1,4} = (-22b_{1})\mu_{0}l\Omega/420 + (-14b_{1})(\mu_{t} - \mu_{0})l\Omega/840 \]

\[ G_{1,8} = (54b_{1})\mu_{0}l\Omega/420 + (54b_{1})(\mu_{t} - \mu_{0})l\Omega/840 \]

\[ G_{1,9} = (63b_{2})\mu_{0}l\Omega/420 + (56b_{2})(\mu_{t} - \mu_{0})l\Omega/840 \]

\[ G_{1,10} = (13b_{1})\mu_{0}l\Omega/420 + (12b_{1})(\mu_{t} - \mu_{0})l\Omega/840 \]
\[
G_{2,3} = (147b_3)\mu_0 l \Omega / 420 + (70b_3)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{2,5} = (-22lb_1)\mu_0 l \Omega / 420 + (-14lb_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{2,7} = (-54b_1)\mu_0 l \Omega / 420 + (-54b_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{2,9} = (63b_3)\mu_0 l \Omega / 420 + (56b_3)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{2,11} = (13lb_1)\mu_0 l \Omega / 420 + (12lb_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{3,4} = (21lb_3)\mu_0 l \Omega / 420 + (14lb_3)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{3,5} = (-21lb_2)\mu_0 l \Omega / 420 + (-14lb_2)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{3,7} = (-63b_2)\mu_0 l \Omega / 420 + (-70b_2)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{3,8} = (-63b_3)\mu_0 l \Omega / 420 + (-70b_3)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{3,10} = (-14lb_3)\mu_0 l \Omega / 420 + (-14lb_3)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{3,11} = (14lb_2)\mu_0 l \Omega / 420 + (14lb_2)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{4,5} = (4l^2b_1)\mu_0 l \Omega / 420 + (3l^2b_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{4,7} = (13lb_1)\mu_0 l \Omega / 420 + (14lb_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{4,9} = (-14lb_3)\mu_0 l \Omega / 420 + (-14lb_3)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{4,11} = (-3l^2b_1)\mu_0 l \Omega / 420 + (-3l^2b_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{5,8} = (13lb_1)\mu_0 l \Omega / 420 + (14lb_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{5,9} = (14lb_2)\mu_0 l \Omega / 420 + (14lb_2)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{5,10} = (3l^2b_1)\mu_0 l \Omega / 420 + (3l^2b_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[
G_{7,8} = (156b_1)\mu_0 l \Omega / 420 + (240b_1)(\mu_l - \mu_0)l \Omega / 840
\]
\[ G_{7,9} = (147b_2)\mu_0l\Omega/420 + (224b_2)(\mu_l - \mu_0)l\Omega/840 \]
\[ G_{7,10} = (22lb_1)\mu_0l\Omega/420 + (30lb_1)(\mu_l - \mu_0)l\Omega/840 \]
\[ G_{8,9} = (147b_3)\mu_0l\Omega/420 + (224b_3)(\mu_l - \mu_0)l\Omega/840 \]
\[ G_{8,11} = (22lb_1)\mu_0l\Omega/420 + (30lb_1)(\mu_l - \mu_0)l\Omega/840 \]
\[ G_{9,10} = (-21b_3)\mu_0l\Omega/420 + (-22lb_3)(\mu_l - \mu_0)l\Omega/840 \]
\[ G_{9,11} = (21lb_2)\mu_0l\Omega/420 + (22lb_2)(\mu_l - \mu_0)l\Omega/840 \]
\[ G_{10,11} = (4l^2b_1)\mu_0l\Omega/420 + (5l^2b_1)(\mu_l - \mu_0)l\Omega/840 \]

### A.4 Spin Stiffness Matrix

The element spin-stiffness matrix \([K_\Omega]\) is symmetric, with the elements:

\[ K_{\Omega,1,1} = (156a_{11})\mu_0l\Omega^2/420 + (72a_{11})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,1,2} = (156a_{12})\mu_0l\Omega^2/420 + (72a_{12})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,1,3} = (147a_{13})\mu_0l\Omega^2/420 + (70a_{13})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,1,4} = (-22la_{12})\mu_0l\Omega^2/420 + (-14la_{12})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,1,5} = (22la_{11})\mu_0l\Omega^2/420 + (14la_{11})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,1,7} = (54a_{11})\mu_0l\Omega^2/420 + (54a_{11})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,1,8} = (54a_{12})\mu_0l\Omega^2/420 + (54a_{12})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,1,9} = (63a_{13})\mu_0l\Omega^2/420 + (56a_{13})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,1,10} = (13la_{12})\mu_0l\Omega^2/420 + (12la_{12})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{1,11}} = (-13l_{a_{11}})\mu_0l\Omega^2/420 + (-12l_{a_{11}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,2}} = (156a_{22})\mu_0l\Omega^2/420 + (72a_{22})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,3}} = (147a_{23})\mu_0l\Omega^2/420 + (70a_{23})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,4}} = (-22l_{a_{22}})\mu_0l\Omega^2/420 + (-14l_{a_{22}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,5}} = (22l_{a_{12}})\mu_0l\Omega^2/420 + (14l_{a_{12}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,7}} = (54a_{12})\mu_0l\Omega^2/420 + (54a_{12})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,8}} = (54a_{22})\mu_0l\Omega^2/420 + (54a_{22})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,9}} = (63a_{23})\mu_0l\Omega^2/420 + (56a_{23})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,10}} = (13l_{a_{22}})\mu_0l\Omega^2/420 + (12l_{a_{22}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{2,11}} = (-13l_{a_{12}})\mu_0l\Omega^2/420 + (-12l_{a_{12}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{3,3}} = (140a_{33})\mu_0l\Omega^2/420 + (70a_{33})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{3,4}} = (-21l_{a_{23}})\mu_0l\Omega^2/420 + (-14l_{a_{23}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{3,5}} = (21l_{a_{13}})\mu_0l\Omega^2/420 + (14l_{a_{13}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{3,7}} = (63a_{13})\mu_0l\Omega^2/420 + (70a_{13})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{3,8}} = (63a_{23})\mu_0l\Omega^2/420 + (70a_{23})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{3,9}} = (70a_{33})\mu_0l\Omega^2/420 + (70a_{33})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{3,10}} = (14l_{a_{23}})\mu_0l\Omega^2/420 + (14l_{a_{23}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{3,11}} = (-14l_{a_{13}})\mu_0l\Omega^2/420 + (-14l_{a_{13}})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega_{4,4}} = (4l^2a_{22})\mu_0l\Omega^2/420 + (3l^2a_{22})(\mu_l - \mu_0)l\Omega^2/840 \]
\[ K_{\Omega,5} = (-4l^2a_{12})\mu_0 l \Omega^2 / 420 + (-3l^2a_{12})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,7} = (-13l a_{12})\mu_0 l \Omega^2 / 420 + (-14la_{12})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,8} = (-13l a_{22})\mu_0 l \Omega^2 / 420 + (-14la_{22})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,9} = (-14la_{23})\mu_0 l \Omega^2 / 420 + (-14la_{23})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,10} = (-3l^2a_{22})\mu_0 l \Omega^2 / 420 + (-3l^2a_{22})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,11} = (3l^2a_{12})\mu_0 l \Omega^2 / 420 + (3l^2a_{12})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,5,5} = (4l^2a_{11})\mu_0 l \Omega^2 / 420 + (3l^2a_{11})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,5,7} = (13l a_{11})\mu_0 l \Omega^2 / 420 + (14la_{11})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,5,8} = (13l a_{12})\mu_0 l \Omega^2 / 420 + (14la_{12})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,5,9} = (14la_{13})\mu_0 l \Omega^2 / 420 + (14la_{13})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,5,10} = (3l^2a_{12})\mu_0 l \Omega^2 / 420 + (3l^2a_{12})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,5,11} = (-3l^2a_{11})\mu_0 l \Omega^2 / 420 + (-3l^2a_{11})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,7,7} = (156a_{11})\mu_0 l \Omega^2 / 420 + (240a_{11})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,7,8} = (156a_{12})\mu_0 l \Omega^2 / 420 + (240a_{12})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,7,9} = (147a_{13})\mu_0 l \Omega^2 / 420 + (224a_{13})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,7,10} = (22la_{12})\mu_0 l \Omega^2 / 420 + (30la_{12})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,7,11} = (-22la_{11})\mu_0 l \Omega^2 / 420 + (-30la_{11})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,8,8} = (156a_{22})\mu_0 l \Omega^2 / 420 + (240a_{22})(\mu_l - \mu_0) l \Omega^2 / 840 \]
\[ K_{\Omega,8,9} = (147a_{23})\mu_0 l \Omega^2 / 420 + (224a_{23})(\mu_l - \mu_0) l \Omega^2 / 840 \]
A.5 Axial Force Stiffness Matrix

The element axial force stiffness matrix \([K_g]\) is symmetric, with the elements:

\[
\begin{align*}
K_{\Omega_{8,10}} &= (22l a_{22}) \mu_0 l \Omega^2 / 420 + (30l a_{22}) (\mu_l - \mu_0) l \Omega^2 / 840 \\
K_{\Omega_{8,11}} &= (-22l a_{12}) \mu_0 l \Omega^2 / 420 + (-30l a_{12}) (\mu_l - \mu_0) l \Omega^2 / 840 \\
K_{\Omega_{9,9}} &= (140 a_{33}) \mu_0 l \Omega^2 / 420 + (210 a_{33}) (\mu_l - \mu_0) l \Omega^2 / 840 \\
K_{\Omega_{9,10}} &= (21l a_{23}) \mu_0 l \Omega^2 / 420 + (22l a_{23}) (\mu_l - \mu_0) l \Omega^2 / 840 \\
K_{\Omega_{9,11}} &= (-21l a_{13}) \mu_0 l \Omega^2 / 420 + (-22l a_{13}) (\mu_l - \mu_0) l \Omega^2 / 840 \\
K_{\Omega_{10,10}} &= (4l^2 a_{22}) \mu_0 l \Omega^2 / 420 + (5l^2 a_{22}) (\mu_l - \mu_0) l \Omega^2 / 840 \\
K_{\Omega_{10,11}} &= (-4l^2 a_{12}) \mu_0 l \Omega^2 / 420 + (-5l^2 a_{12}) (\mu_l - \mu_0) l \Omega^2 / 840 \\
K_{\Omega_{11,11}} &= (4l^2 a_{11}) \mu_0 l \Omega^2 / 420 + (5l^2 a_{11}) (\mu_l - \mu_0) l \Omega^2 / 840
\end{align*}
\]
\[ K_{g4,4} = (4l^2)F_0/30l - (2l^2)c/60 - (4l^2)dl/420 - (11l^2)e^2/2520 \]

\[ K_{g4,8} = (3l)F_0/30l - (6l)c/60 - (15l)dl/420 - (42l)e^2/2520 \]

\[ K_{g4,10} = (-l^2)F_0/30l - (-l^2)c/60 - (-3l^2)dl/420 - (-11l^2)e^2/2520 \]

\[ K_{g5,5} = (4l^2)F_0/30l - (2l^2)c/60 - (4l^2)dl/420 - (11l^2)e^2/2520 \]

\[ K_{g5,7} = (-3l)F_0/30l - (-6l)c/60 - (-15l)dl/420 - (-42l)e^2/2520 \]

\[ K_{g5,11} = (-l^2)F_0/30l - (-l^2)c/60 - (-3l^2)dl/420 - (-11l^2)e^2/2520 \]

\[ K_{g7,7} = (36)F_0/30l - (36)c/60 - (72)dl/420 - (180)e^2/2520 \]

\[ K_{g7,11} = (-3l)F_0/30l - (0)c/60 - (6l)dl/420 - (30l)e^2/2520 \]

\[ K_{g8,8} = (36)F_0/30l - (36)c/60 - (72)dl/420 - (180)e^2/2520 \]

\[ K_{g8,10} = (3l)F_0/30l - (0)c/60 - (-6l)dl/420 - (-30l)e^2/2520 \]

\[ K_{g10,10} = (4l^2)F_0/30l - (6l^2)c/60 - (18l^2)dl/420 - (65l^2)e^2/2520 \]

\[ K_{g11,11} = (4l^2)F_0/30l - (6l^2)c/60 - (18l^2)dl/420 - (65l^2)e^2/2520 \]

### A.6 Axial Reduction Matrix

The element axial reduction matrix \([\mathbf{K}_{AxRed}]\) is symmetric, with the elements:

\[ K_{AxRed,1} = 36/30l \]

\[ K_{AxRed,5} = 3l/30l \]

\[ K_{AxRed,7} = -36/30l \]

\[ K_{AxRed,11} = 3l/30l \]
\[ K_{AxRed_{2,2}} = \frac{36}{30l} \]

\[ K_{AxRed_{2,4}} = -\frac{3l}{30l} \]

\[ K_{AxRed_{2,8}} = -\frac{36}{30l} \]

\[ K_{AxRed_{2,10}} = -\frac{3l}{30l} \]

\[ K_{AxRed_{4,4}} = \frac{4l^2}{30l} \]

\[ K_{AxRed_{4,8}} = \frac{3l}{30l} \]

\[ K_{AxRed_{4,10}} = -\frac{l^2}{30l} \]

\[ K_{AxRed_{5,5}} = \frac{4l^2}{30l} \]

\[ K_{AxRed_{5,7}} = -\frac{3l}{30l} \]

\[ K_{AxRed_{5,11}} = -\frac{l^2}{30l} \]

\[ K_{AxRed_{7,7}} = \frac{36}{30l} \]

\[ K_{AxRed_{7,11}} = -\frac{3l}{30l} \]

\[ K_{AxRed_{8,8}} = \frac{36}{30l} \]

\[ K_{AxRed_{8,10}} = \frac{3l}{30l} \]

\[ K_{AxRed_{10,10}} = \frac{4l^2}{30l} \]

\[ K_{AxRed_{11,11}} = \frac{4l^2}{30l} \]
Appendix B

CurveFEM Program Structure

The main program is CurveFEM.f90 and calls several subroutines to build the finite element model and solve the eigenvalue problem. The input is a file with blade properties as in the FAST blade data file, along with blade geometry, operating speed, and output requests. The output is the natural frequencies (square roots of the eigenvalues) and selected eigenvectors (mode shapes). The program calls the following subroutines:

- read_input.f90
- taper_stiff_db1.f90
- taper_mass_db1.f90
- taper_frame_spin_db1.f90
  - taper_axial_force_db1.f90
- band2full_db1.f90
- decomp_full_db1.f90
- gauss_db1.f90
- jacobi_db1.f90
These subroutines have the prefix *dbl.f90 to distinguish them as using double precision arithmetic. Originally the code was written to solve the eigenvalue problem using the Coriolis matrix as in Meirovitch [42] as described in Chapter 2, which required double precision for accuracy. The program retains double precision arithmetic with the simpler model with the Coriolis terms neglected.

Several of these subroutines call the subroutine matmul_dbl.f90, which performs matrix multiplication.

The subroutine taper_stiff_dbl.f90 builds the banded elastic stiffness matrix for structure with space frame elements with linearly tapered properties. The subroutine first builds local matrices in the local element $L_j$ system. The global matrix is assembled from the element matrices. The global matrix is referenced to the pitched $j$ system.

The subroutine taper_mass_dbl.f90 builds the banded consistent-mass matrix for structure with space frame elements with linearly tapered properties. As in the stiffness subroutine, this subroutine builds local matrices and then assembles the global matrix in the pitched coordinate system.

The subroutine taper_frame_spin_dbl.f90 builds the banded centrifugal force matrix, the gyroscopic (or Coriolis) matrix, and the spin-stiffness matrix for the tapered elements. The algorithm expresses the matrices in the $j$ system, which is rotating at constant speed. For the centrifugal force matrix, the routine calls taper_axial_force_dbl.f90 that computes the axial force at the inboard nodes of the tapered elements.

The subroutine band2full_dbl.f90 converts banded, symmetric matrices into full matrices for the eigenvalue problem. Then the subroutine decomp_full_dbl.f90 decomposes the full matrix into an upper triangle. Next, the subroutine gauss_dbl.f90 finds the inverse of the upper triangle. The program performs the necessary matrix algebra to convert the matrices into the standard eigenvalue problem, which is then solved with the jacobi_dbl.f90 subroutine.
Appendix C

Summary of FAST Modifications

This appendix summarizes the changes to the FAST code source files, subroutines and input files for CurveFAST. The author modified version 6.10a of FAST that used version 12.58 of AeroDyn.

**Source File AeroSubs.f90** In subroutine ReadFF, the author changed the switch for dynamic inflow from 8 m/s to 9 m/s, due to instabilities in the model from 8 to 9 m/s.

**Source File FAST_IO.f90** In several subroutines, the author added changes to read in new parameters in the input files. The primary file had a new switch for the fourth (torsional) blade mode. The blade file had new parameters for the fourth blade mode, and had new input for mode frequencies and eigenvectors. New subroutine CalcT_EP calculated the transformation from the j (pitched) system to the Lj (element) system. New subroutine taper_stiff calculated the elastic stiffness matrix. New subroutine AxRedMatrix calculated the axial reduction matrix.

**Source File FAST Mods.f90** The author added several variables to many of the modules in this file.

**Source File FAST.f90** Many changes in several subroutines for new variables and mode shapes. Changes to subroutine RtHS for blade equations of motion. New subroutine
band2full to transform banded matrix into full matrix.

Source File GenUse.f90 New subroutine matmul for matrix multiplication.