Stiffness Matrices for Trusses

Basic Stiffness

A basic truss member is shown in Figure 1. From equilibrium we know that the axial forces are related by:

\[ f_i = -f_j \]

The total elongation of the member is:

\[ \Delta = u_j - u_i \]

From mechanics of materials you will recall that the axial force and the resulting elongation are related by:

\[ -f_i = f_j = \frac{EA}{L} \Delta \]

Note that the signs must be as shown if we choose tension as positive. If we combine the previous two equations, and write the result in matrix form, we get:

\[
\begin{bmatrix}
  f_i \\
  f_j
\end{bmatrix} =
\begin{bmatrix}
  \frac{EA}{L} & -\frac{EA}{L} \\
  -\frac{EA}{L} & \frac{EA}{L}
\end{bmatrix}
\begin{bmatrix}
  u_i \\
  u_j
\end{bmatrix} = [K_m]
\begin{bmatrix}
  u_i \\
  u_j
\end{bmatrix}
\]

which is the basic stiffness relationship and \([K_m]\) is the local stiffness matrix. As an example, suppose the \(i\)-end of the member is fixed and the displacement at the \(j\)-end is 0.05" to the right. If the member properties are: \(A = 2 \text{ in}^2\), \(E = 29000 \text{ ksi}\), \(L = 10 \text{ ft}\), then the above equation becomes:

\[
\begin{bmatrix}
  f_i \\
  f_j
\end{bmatrix} =
\begin{bmatrix}
  483.333 & -483.333 \\
  -483.333 & 483.333
\end{bmatrix}
\begin{bmatrix}
  0 \\
  0.05
\end{bmatrix}
\]

Solving for the forces, we get:

\[
\begin{bmatrix}
  f_i \\
  f_j
\end{bmatrix} = \begin{bmatrix}
  -24.16 \\
  24.16
\end{bmatrix} \text{ kip}
\]

Thus we see the member is in tension.
General Stiffness

In the case we just looked at, the member was horizontal. We need to look at the more general case where the member is oriented at some arbitrary angle to the horizontal. We will refer to the coordinate system which has the x-axis along the member as the **local axis system**. The local axis system can have any orientation with respect to the **global axis system**, designated by X-Y in Figure 2.

In the global axis system there will be two force components and two displacement components at either end of the member. Figures 3 and 4 show the relationships between the components in local coordinates and those in global coordinates.

At either end, the forces are related by:

\[ F_x = f \cos \theta \quad F_y = f \sin \theta \]

The displacement along the member axis is \( e \) and is made up of two contributions:

\[ e = u \cos \theta + v \sin \theta \]

Where \( u \) and \( v \) are the components of displacement in global coordinates.

If we write the above relations in matrix form and including both ends, we get:

\[
\begin{bmatrix}
F_{x_i} \\
F_{y_i} \\
F_{x_j} \\
F_{y_j}
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & 0 \\
\sin \theta & 0 \\
0 & \cos \theta \\
0 & \sin \theta
\end{bmatrix}
\begin{bmatrix}
f_i \\
f_j
\end{bmatrix}
\begin{bmatrix}
e_i \\
e_j
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_i \\
u_j \\
v_j
\end{bmatrix}
\]
Now we can make the following definition of a **transformation matrix**, literally, a matrix which transforms a vector from one coordinate system to another.

\[
[T] = \begin{bmatrix}
\cos \theta & 0 \\
\sin \theta & 0 \\
0 & \cos \theta \\
0 & \sin \theta
\end{bmatrix}
\]

In this instance \([T]\) transforms a vector in local coordinates to a vector in global coordinates. The force and displacement relationships then become:

\[
\begin{bmatrix}
F_{xi} \\
F_{yi} \\
F_{xj} \\
F_{yj}
\end{bmatrix} = [T] \begin{bmatrix}
f_i \\
f_j \\
e_i \\
e_j
\end{bmatrix} = [T] \begin{bmatrix}
u_i \\
v_j \\
u_j \\
v_j
\end{bmatrix}
\]

If we substitute the above relations into the local stiffness relation, we get:

\[
\begin{bmatrix}
F_{xi} \\
F_{yi} \\
F_{xj} \\
F_{yj}
\end{bmatrix} = [T] K_m [T]^T \begin{bmatrix}
u_i \\
v_j \\
u_j \\
v_j
\end{bmatrix}
\]

Then we define the global stiffness matrix \([K]\) as:

\[
[K] = [T][K_m][T]^T
\]

where

\[
[K] = \frac{EA}{L} \begin{bmatrix}
\cos^2 \theta & \sin \theta \cos \theta & -\cos^2 \theta & -\sin \theta \cos \theta \\
\sin \theta \cos \theta & \sin^2 \theta & -\sin \theta \cos \theta & -\sin^2 \theta \\
-\cos^2 \theta & -\sin \theta \cos \theta & \cos^2 \theta & \sin \theta \cos \theta \\
-\sin \theta \cos \theta & -\sin^2 \theta & \sin \theta \cos \theta & \sin^2 \theta
\end{bmatrix}
\]