Multiple Degree of Freedom Systems (MDOF)

Consider the system shown. The equations of motion are:

\[-k_1 x_1 + k_2 (x_2 - x_1) = m_1 \ddot{x}_1\]
\[-k_2 (x_2 - x_1) = m_2 \ddot{x}_2\]

Or in matrix form:

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2 \\
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2 \\
\end{bmatrix}
+
\begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2 \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
\end{bmatrix}
=
\begin{bmatrix}
  0 \\
  0 \\
\end{bmatrix}
\]

We then guess a solution of the form:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
\end{bmatrix}
=
\begin{bmatrix}
  X_1 \\
  X_2 \\
\end{bmatrix}
\cos \omega t
\]

Note that we could equally easily use a sine function. Substituting into the equations of motion gives:

\[
-\omega^2
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2 \\
\end{bmatrix}
\begin{bmatrix}
  X_1 \\
  X_2 \\
\end{bmatrix}
+
\begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2 \\
\end{bmatrix}
\begin{bmatrix}
  X_1 \\
  X_2 \\
\end{bmatrix}
=
\begin{bmatrix}
  0 \\
  0 \\
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
  k_1 + k_2 - m_1 \omega^2 & -k_2 \\
  -k_2 & k_2 - m_2 \omega^2 \\
\end{bmatrix}
\begin{bmatrix}
  X_1 \\
  X_2 \\
\end{bmatrix}
=
\begin{bmatrix}
  0 \\
  0 \\
\end{bmatrix}
\]

The above equation has the obvious solution \( X_1 = X_2 = 0 \). However this solution is not of much interest and is referred to as the trivial solution. Another possibility (remember our discussion of Cramer’s rule) is that:

\[
\det
\begin{bmatrix}
  k_1 + k_2 - m_1 \omega^2 & -k_2 \\
  -k_2 & k_2 - m_2 \omega^2 \\
\end{bmatrix}
= 0
\]

That is, the determinant of the matrix is zero. For example let’s look at the case where \( k_1 = k_2 = k \) and \( m_1 = m_2 = m \). Evaluating the determinant leads us to the characteristic equation:

\[
(2k - m \omega^2)(k - m \omega^2) - k^2 = 0
\]

If we solve this equation for \( \omega^2 \), we get:

\[
\omega^2 = \frac{3 \pm \sqrt{5}}{2} \frac{k}{m}
\]
The above roots of the characteristic equation are the eigenvalues of the equation and are physically the possible natural frequencies of the system. Associated with each possible natural frequency (or eigenvalue) is a mode shape (or eigenvector). The eigenvectors can be determined by substituting the eigenvalues back into the equation for $X_1$ and $X_2$. Thus we get:

$$
\phi_1 = \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_1 = \begin{pmatrix} 1 + \sqrt{5} \\ 2 \end{pmatrix} \\
\phi_2 = \begin{pmatrix} \phi_{12} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_2 = \begin{pmatrix} 1 - \sqrt{5} \\ 2 \end{pmatrix}
$$

The above eigenvectors (mode shapes) have been normalized so that the displacement of the first mass is set to 1 in each case. Thus they would be referred to as normal modes. In addition the first mode shape $\phi_1$ is referred to as the fundamental.

The shapes are as shown below:

![Mode 1 and Mode 2](image)

The significance of these shapes is that they represent the vibrations which will occur if the system is started from rest. Thus if we begin with the masses held so that they are on the same side of the equilibrium position with the displacement of the top mass 1.618 times that of the bottom mass, the masses will vibrate at a natural frequency of $0.618 \sqrt{\frac{k}{m}}$ and the ratio of the displacement of the top mass to that of the bottom mass will always be 1.618.

Of course all real situations will be such that the motions of the masses will be some combination of the two modes. For example if we move the top mass 1 unit from equilibrium without moving the bottom mass and release the system from rest, we get the solution:

$$
x_1 = \frac{1}{\sqrt{5}} \cos \omega_1 t - \frac{1}{\sqrt{5}} \cos \omega_2 t
$$

$$
x_2 = \frac{1 + \sqrt{5}}{2\sqrt{5}} \cos \omega_1 t - \frac{1 - \sqrt{5}}{2\sqrt{5}} \cos \omega_2 t
$$
In terms of the eigenvectors $\phi_1$ and $\phi_2$, the above equation can be written:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \{ \phi_1 \} \cos \omega_1 t - \frac{1}{\sqrt{5}} \{ \phi_2 \} \cos \omega_2 t$$

Thus we see that the final displacement history is a linear combination of the two modes, i.e. $1/\sqrt{5}$ times $\phi_1$ minus $1/\sqrt{5}$ times $\phi_2$.

**Principal coordinates**

The above example suggests that the general solution to the 2DOF case we have been discussing is given by:

$$x_1 = \phi_{11} q_1 + \phi_{12} q_2$$
$$x_2 = \phi_{21} q_1 + \phi_{22} q_2$$

Where $q_1$ and $q_2$ are given by:

$$q_1 = A_1 \sin \omega_1 t + B_1 \cos \omega_1 t$$
$$q_2 = A_2 \sin \omega_2 t + B_2 \cos \omega_2 t$$

The equation for the displacements can be rewritten in matrix form as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \text{ or } \{ x \} = [\Phi] \{ q \}$$

The matrix $[\Phi]$ is then a coordinate transformation matrix which maps the original coordinates $\{ x \}$ into the **principal coordinates** $\{ q \}$. The inverse transformation is

$$\{ q \} = [\Phi]^{-1} \{ x \}$$

However it turns out that due to the properties of the eigenvectors, $[\Phi]^T = [\Phi]^{-1}$, and the above relationship becomes:

$$\{ q \} = [\Phi]^T \{ x \}$$

If we substitute the transformation relationship into the equations of motion, and pre-multiply by $[\Phi]^T$, we get:

$$[\Phi]^T [m] [\Phi] \{ q \} + [\Phi]^T [k] [\Phi] \{ q \} = \{ 0 \}$$
To explain why we went through the above process, let’s use the 2DOF case we’ve been considering. If we perform the matrix operations indicated, we get:

\[
\begin{bmatrix}
\frac{1}{1 + \sqrt{5}} & \frac{1}{1 - \sqrt{5}} \\
\frac{1}{2} & \frac{2}{2}
\end{bmatrix}^T
\begin{bmatrix}
m & 0 \\
0 & m
\end{bmatrix}
\begin{bmatrix}
\frac{1}{1 + \sqrt{5}} & \frac{1}{1 - \sqrt{5}} \\
\frac{1}{2} & \frac{2}{2}
\end{bmatrix}
= \begin{bmatrix}
\frac{5 + \sqrt{5}}{2}m & 0 \\
0 & \frac{5 - \sqrt{5}}{2}m
\end{bmatrix}
= \begin{bmatrix}
M_1 & 0 \\
0 & M_2
\end{bmatrix}
\]

The coefficients \(M_1\) and \(M_2\) are called the **modal masses** for modes 1 and 2 respectively.

If we look at the stiffness coefficients, we get:

\[
\begin{bmatrix}
\frac{1}{1 + \sqrt{5}} & \frac{1}{1 - \sqrt{5}} \\
\frac{1}{2} & \frac{2}{2}
\end{bmatrix}^T
\begin{bmatrix}
2k & -k \\
-k & k
\end{bmatrix}
\begin{bmatrix}
\frac{1}{1 + \sqrt{5}} & \frac{1}{1 - \sqrt{5}} \\
\frac{1}{2} & \frac{2}{2}
\end{bmatrix}
= \begin{bmatrix}
\frac{5 - \sqrt{5}}{2}k & 0 \\
0 & \frac{5 + \sqrt{5}}{2}k
\end{bmatrix}
= \begin{bmatrix}
K_1 & 0 \\
0 & K_2
\end{bmatrix}
\]

The most important feature of the above two equations is that the resulting matrices are diagonal matrices and the transformed equations of motion are now:

\[
[M][\ddot{q}] + [K][q] = \{0\}
\]

or

\[
M_1\ddot{q}_1 + K_1q_1 = 0 \\
M_2\ddot{q}_2 + K_2q_2 = 0
\]

These equations are uncoupled, i.e., each contains only one dependent variable \(q\). If each equation is divided by the modal mass, the above equations can be rewritten as:

\[
\ddot{q}_1 + \omega_1^2q_1 = 0 \\
\ddot{q}_2 + \omega_2^2q_2 = 0
\]

And we see that each equation has the same form as the SDOF system we considered earlier.

This outcome is the result of the orthogonality of the mode shapes. For our 2DOF system:

\[
\{\phi_1\}^T[m]\{\phi_2\} = \{\phi_2\}^T[m]\{\phi_1\} = \{0\}
\]

\[
\{\phi_1\}^T[k]\{\phi_2\} = \{\phi_2\}^T[k]\{\phi_1\} = \{0\}
\]

\[
M_i = \{\phi_i\}^T[m]\{\phi_i\} \quad K_i = \{\phi_i\}^T[k]\{\phi_i\}
\]