Earthquake response (MDOF)

Consider the system shown. The equations of motion are:

\[-k_1(x_1 - x_e) + k_2(x_2 - x_1) = m_1\ddot{x}_1\]
\[-k_2(x_2 - x_1) = m_2\ddot{x}_2\]

The relative displacements are, \(x_{r1} = x_1 - x_e\) and \(x_{r2} = x_2 - x_e\). If we write the above equations in terms of relative displacements:

\[-k_1x_{r1} + k_2(x_{r2} - x_{r1}) = m_1\ddot{x}_{r1} + m_1\ddot{x}_e\]
\[-k_2(x_{r2} - x_{r1}) = m_2\ddot{x}_{r2} + m_2\ddot{x}_e\]

Or in matrix form:

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2 \\
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_{r1} \\
  \ddot{x}_{r2} \\
\end{bmatrix} +
\begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2 \\
\end{bmatrix}
\begin{bmatrix}
  x_{r1} \\
  x_{r2} \\
\end{bmatrix} =
\begin{bmatrix}
  -m_1\ddot{x}_e \\
  -m_2\ddot{x}_e \\
\end{bmatrix}
\]

The right side can be written as:

\[
\begin{bmatrix}
  -m_1\ddot{x}_e \\
  -m_2\ddot{x}_e \\
\end{bmatrix} =
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2 \\
\end{bmatrix}
\begin{bmatrix}
  1 \\
  1 \\
\end{bmatrix}\ddot{x}_e =
\begin{bmatrix}
  m_1 \ddot{x}_e \\
  m_2 \ddot{x}_e \\
\end{bmatrix}
\]

Thus we have in general:

\[
[m][\ddot{x}_r] + [k][x_r] = -[m][1]\ddot{x}_e
\]

Now we use the coordinate transformation matrix \([\Phi]\) to transform the original coordinates \([x]\) into the principal coordinates \([q]\). If we substitute the transformation relationship into the equations of motion, and pre-multiply by \([\Phi]^T\), we get:

\[
[\Phi]^T [m][\Phi][\ddot{q}] + [\Phi]^T [k][\Phi][q] = -[\Phi]^T [m][1]\ddot{x}_e
\]

Or

\[
[M][\ddot{q}] + [K][q] = -[\Phi]^T [m][1]\ddot{x}_e
\]

Since \([M]\) and \([K]\) are both diagonal matrices, the above system can be written as a series of uncoupled equations. For our example:

\[
M_1\ddot{q}_1 + K_1q_1 = -[\phi_1]^T [m][1]\ddot{x}_e
\]
\[
M_2\ddot{q}_2 + K_2q_2 = -[\phi_2]^T [m][1]\ddot{x}_e
\]
If we divide both sides of each of the above equations by the respective modal mass $M_i$, we get

\[
\ddot{q}_1 + \omega_1^2 q_1 = -\frac{\phi_1^T Y [m] \dot{\xi}_e}{M_1} \\
\ddot{q}_2 + \omega_2^2 q_2 = -\frac{\phi_2^T Y [m] \dot{\xi}_e}{M_2}
\]

Now we can define the \textbf{modal earthquake participation factor} $\mu_i$:

\[
\mu_i = \frac{\phi_i^T Y [m] \dot{\xi}_e}{M_i}
\]

And the equations become:

\[
\ddot{q}_1 + \omega_1^2 q_1 = -\mu_1 \ddot{\xi}_e \\
\ddot{q}_2 + \omega_2^2 q_2 = -\mu_2 \ddot{\xi}_e
\]

Each of the above equations is a SDOF equation with only one dependent variable. To evaluate the earthquake response, we can use response spectra. Thus the spectral displacements corresponding to each mode are obtained by computing the modal period and then reading the value of $S_{di}$ from the spectrum.

Once we know all the values of $S_{di}$, we can say that the responses $q_i$ are given by:

\[
q_i = \mu_i S_{di}
\]

We can then get the response in terms of the actual relative displacements, $x_r$, as:

\[
\{x_r\} = [\Phi]\{\mu S_d\}
\]

Where $[\mu S_d]$ is a diagonal matrix of the spectral responses.

Now let’s look at the two degree of freedom example we have been working on. Let $k = 20 \text{ kip/in}$ and $m = 0.25 \text{ kip·sec}^2/\text{in}$. The natural frequencies and associated periods are therefore:

\[
\omega_1 = 5.528 \text{ rad/sec} \quad T_1 = 1.137 \text{ sec} \\
\omega_2 = 14.472 \text{ rad/sec} \quad T_2 = 0.434 \text{ sec}
\]

Using the UBC response spectrum with $C_a = 0.4$ and $C_v = 0.56$ gives:

\[
S_{d1} = 6.222 \text{ in} \quad \quad S_{d2} = 1.845 \text{ in}
\]
The two normal mode shapes are:

\[ \{ \phi_1 \} = \begin{bmatrix} 1 \\ 1.618 \end{bmatrix} \quad \{ \phi_2 \} = \begin{bmatrix} 1 \\ -0.618 \end{bmatrix} \]

Thus the modal masses are:

\[ M_1 = \begin{bmatrix} 1 \\ 1.618 \end{bmatrix}^T \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1.618 \end{bmatrix} = 0.904 \]

\[ M_2 = \begin{bmatrix} 1 \\ -0.618 \end{bmatrix}^T \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ -0.618 \end{bmatrix} = 0.345 \]

The earthquake modal participation factors are:

\[ \mu_1 = \frac{1}{0.904} \begin{bmatrix} 1 \\ 1.618 \end{bmatrix}^T \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1.618 \end{bmatrix} = 0.723 \]

\[ \mu_2 = \frac{1}{0.345} \begin{bmatrix} 1 \\ -0.618 \end{bmatrix}^T \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ -0.618 \end{bmatrix} = 0.277 \]

You should note that \( \mu_1 + \mu_2 = 1 \). This property is useful as a check, but you must remember that it is only valid if the modes you use are normal modes.

Now to continue with our example. The values of \( x_r \) that are obtained from the transformation equation are:

\[ [x_r] = \begin{bmatrix} 1 \\ 1.618 \end{bmatrix} \begin{bmatrix} 0.723 \cdot 6.222 & 0 \\ 0 & 0.277 \cdot 1.845 \end{bmatrix} = \begin{bmatrix} 4.499 \\ 7.279 \end{bmatrix} \]

The first row of the above final matrix represents the displacements at the first floor due to each mode, The second row represents the displacements at the roof. The actual displacement at the first floor is of course a time-varying quantity. The values above are the maximum values which occur due to each mode. Thus we usually add them by a sum-of-squares procedure. The values are:

\[ x_{r1} = \sqrt{4.499^2 + 0.511^2} = 4.528 \text{ in} \]

\[ x_{r2} = \sqrt{7.279^2 + (-0.316)^2} = 7.286 \text{ in} \]
Base Shear and Story Shears for MDOF systems

We now must determine the forces at each level. The spectral acceleration corresponding to mode $i$ is:

$$|\ddot{q}_i| = \mu_i \omega_i^2 S_{di}$$

In terms of the original displacement coordinate $x_r$ at floor level $k$:

$$|\ddot{x}_{rk}| = \phi_{ki} \mu_i \omega_i^2 S_{di}$$

Where the $ki$ subscripts denotes the contribution at level $k$ due to mode $i$. From Newton’s second law, the inertial force at level $k$ is:

$$F_{ki} = m_k \phi_{ki} \mu_i \omega_i^2 S_{di}$$

And the base shear for mode $i$ would be:

$$V_i = F_{1i} + F_{2i} + F_{3i} + \cdots = \mu_i \omega_i^2 S_{di} \sum m_k \phi_{ki}$$

Or, given the definition of the modal participation factor $\mu_i$,

$$V_i = M_i \mu_i^2 \omega_i^2 S_{di}$$

We can now write the story forces for mode $i$, $F_{ki}$, as

$$F_{ki} = \frac{m_k \phi_{ki} V_i}{M_i \mu_i} = \frac{m_k \phi_{ki}}{\sum m_k \phi_{ki}} V_i$$