Statics Review

(1) Introduction

Statics is concerned with the equilibrium of bodies under the action of forces. Statics is used to determine the reactive and internal forces that a member experiences when subjected to applied loads. Also, properties such as centroids, centers of gravity, moments of inertia, etc. are considered in engineering statics.

(2) Force

A force represents the action of one body upon another. Forces can be represented mathematically with vectors: (i) the length of the vector represents the magnitude of the force, the orientation of the vector represents the direction of the force (line of action and sense), and the tail of the vector represents the point where the force is applied.

In vector form (3–D):
\[ F = F_x i + F_y j + F_z k \]

(3) Resolution of a Force (3–D)

Given a force with its line of action defined by the origin and a point \((x, y, z)\), we can separate it into components as follows:

Length of line between origin and \((x, y, z)\):
\[ R = \sqrt{x^2 + y^2 + z^2} \]

Component magnitudes:
\[
F_x = \frac{x}{R} F \\
F_y = \frac{y}{R} F \\
F_z = \frac{z}{R} F
\]

Example 1: Resolve the 150 lb force into two components, one along line \(P\) the other along line \(Q\).

\[
F = 150 lb \\
F \sin 25^\circ = F_Q \cos 70^\circ \\
F_Q = \frac{150 lb \cos 25^\circ}{\sin 70^\circ} = 150 lb \cdot \frac{\cos 25^\circ}{\sin 70^\circ} \\
F_Q = 67.5 lb \\
F_p + F_Q \cos 70^\circ = F \sin 25^\circ \\
F_p = 150 lb \sin 25^\circ - 67.5 lb \cos 70^\circ \\
F_p = 113 lb
\]
Resultant

- N number of forces can be added to arrive at a single force, the resultant of these forces, \( F = \sum F_n \).

In 2-D

Vector components:
\[
F_x = \sum_{i=1}^{n} F_{xi}, \quad F_y = \sum_{i=1}^{n} F_{yi}
\]

Magnitude of resultant:
\[
F = \sqrt{\left(\sum_{i=1}^{n} F_{xi}\right)^2 + \left(\sum_{i=1}^{n} F_{yi}\right)^2}
\]

Direction of resultant wrt x:
\[
\theta = \arctan\left(\frac{F_y}{F_x}\right)
\]

Force polygon (2-D)

Example 2: Find the resultant of the following system of forces and its direction.

Summary:

\[
\begin{array}{ccc}
\text{Force} & F_{xi} (lb) & F_{yi} (lb) \\
F_1 & +129.9 & +75 \\
F_2 & -27.4 & +75.2 \\
F_3 & 0 & -110 \\
F_4 & +96.6 & -25.9 \\
F_x & =199.1 lb & F_y = 14.3 lb \\
F = \sqrt{F_x^2 + F_y^2} = 199.6 lb \\
\tan \alpha = \frac{F_y}{F_x} = \frac{14.3}{199.1} = 0.0716 \\
\alpha = 4.1^\circ
\end{array}
\]
(5) Moments

- A moment represents the tendency of a force to rotate a rigid body about a pivot axis. Moments can also be represented mathematically with vectors. Moment vector for a force $\mathbf{F}$ about a point $O$ is given by the cross-product of $\mathbf{F}$ and the vector from point $O$ to the point where $\mathbf{F}$ is applied, $\mathbf{r}$.

  **Moment vector:** $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$

  **Moment components:**
  - $M_x = yF_z - zF_y$
  - $M_y = zF_x - xF_z$
  - $M_z = xF_y - yF_x$

  **Moment magnitude:** $M_O = |\mathbf{r}| |\mathbf{F}| \sin \theta$

- Couple: a system of two equal-magnitude forces, parallel to each other and with opposite sense.

  The two forces $\mathbf{F}$ can be replaced by a moment $\mathbf{M}_O$ with magnitude $M_O = dF$

  This moment $\mathbf{M}_O$ can be moved to any location on the body.

The concept of the couple allows us to move a force $\mathbf{F}$ a distance $d$ from the original point of application and replace it with a force $\mathbf{F}$ and a couple $\mathbf{M}$ of magnitude $Fd$ at this new location. Alternatively, a force-couple system can be replaced by a single force by locating the force at a distance $d = M/F$ from the point where the force-couple system is applied.

(6) Equivalent Systems of Forces (Rigid Bodies)

Two systems of forces, $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$ and $\mathbf{F}_1', \mathbf{F}_2', \mathbf{F}_3', \ldots$, are said to be equivalent if, and only if,

$$\sum \mathbf{F} = \sum \mathbf{F}' \quad \text{and} \quad \sum \mathbf{M}_o = \sum \mathbf{M}_o$$

**Example 3:** Reduce the following force system to a single force.
Example 3: cont.

Step 1) Force rectangular components

![Diagram](image)

Step 2) Sum force components

\[ \sum F_x = 100\, \text{lb} + 141.4\, \text{lb} - 43.3\, \text{lb} = 198.1\, \text{lb} \rightarrow \]
\[ \sum F_y = 141.4\, \text{lb} - 150.0\, \text{lb} - 25.0\, \text{lb} = 33.6\, \text{lb} \downarrow \]

Step 3) Determine the magnitude of the resultant \( R \)

\[
R = \sqrt{(198.1\, \text{lb})^2 + (33.6\, \text{lb})^2} = 200.9\, \text{lb}
\]

Step 4) Determine the location of the resultant \( R \)

\[
\alpha = \arctan\left(\frac{33.6\, \text{lb}}{198.1\, \text{lb}}\right) = 9.63^\circ
\]
\[
\sum M_A = -141.4\, \text{lb}(1\, \text{in}) + 25\, \text{lb}(2\, \text{in}) + 43.3\, \text{lb}(1\, \text{in}) = 48.1\, \text{in}\cdot\text{lb}
\]
\[
d = \frac{\sum M_A}{R} = \frac{48.1\, \text{in}\cdot\text{lb}}{200.9\, \text{lb}} = 0.239\, \text{in}
\]

Summary

![Diagram](image)

(7) Particle Equilibrium

A force system on a particle can be reduced to a single force, the resultant \( F = \sum F_n \). When \( F \) is equal to zero, the particle is said to be in equilibrium, and \( \sum F_n = 0 \).

In 2-D

Vector components: \( \sum F_x = 0 \) and \( \sum F_y = 0 \) Two equations imply that we can solve for two unknowns.
Example 4: Determine the horizontal and vertical components of the reaction at the pin.

\[ F_1 = 100 \text{ lb} \]

\[ F_{1x} = 100 \text{ lb} \cos 30^\circ = 86.6 \text{ lb} \rightarrow \]

\[ F_{1y} = 100 \text{ lb} \sin 30^\circ = 50 \text{ lb} \uparrow \]

Apply equilibrium:

\[ \sum F_x = 0; \quad 86.6lb + 200lb + R_x = 0 \quad \Rightarrow \quad R_x = -286.6lb, \text{ or } R_x = 286.6lb \leftarrow \]

\[ \sum F_y = 0; \quad 50lb + R_y = 0 \quad \Rightarrow \quad R_y = -50.0lb, \text{ or } R_y = 50.0lb \downarrow \]

(8) Equilibrium of a Rigid Body

Rigid bodies can support forces and moments; therefore, the force system for a rigid body consists of

\[ F = \sum F_n \text{ and } M = \sum (\mathbf{r}_n \times F_n). \]

When \( F \) and \( M \) are zero, the rigid body is said to be in equilibrium.

\[ \sum F_n = 0 \text{ and } \sum (\mathbf{r}_n \times F_n) = 0 \]

These vector equations are equivalent to six scalar equations in 3-D.

In 2-D

Components: \( \sum F_x = 0, \sum F_y = 0, \) and \( \sum M = 0 \)

Three equations imply that we can solve for three unknowns.

Static equilibrium problems usually involve solving for unknown reactions and internal forces. To determine these unknown quantities using the equilibrium equation, a Free-Body-Diagram (FBD) is usually employed. A FBD represents a body in equilibrium, depicting all applied loads, moments, and unknown reactions. To determine internal forces (such as the tension in a cable), a portion of the body can be conceptually removed and replaced by the forces and moments this portion imparts to the body.

Example 5: Draw the FBD for the following cases:

**space diagrams**

**free-body-diagrams**
• Supports and corresponding reactions

- Roller: one force reaction
- Pin: two force reactions
- Built-in (fixed): two force reactions and one moment reaction
- Free end: zero reactions

• Rigid bodies subjected to concurrent forces

A system of forces is said to be concurrent if the lines of action of all the forces in question pass through a single point. From an equilibrium standpoint, this case is identical to a particle. That is, only force equilibrium is necessary to ensure equilibrium of the body, in 2-D two equilibrium equations and in 3-D three equilibrium equations.

• Determinacy

A rigid body is said to be statically determinate if the number of unknown quantities is equal to the number of equations of static equilibrium; e.g., in 2-D we have three equations, so we can at most solve for three unknown quantities. Problems with more unknowns than equations of static equilibrium are said to be statically indeterminate – the number of extra unknowns is called the degree of indeterminacy, and each of the unknowns is called a redundant. Those problems with less unknowns than equations are said to be unstable.

In the analysis of structures, determinacy is divided into external determinacy and internal determinacy. Structures that appear to be statically indeterminate externally, may be determinate internally. That is, after dividing the body into a number of components and applying three equations of equilibrium to each component, the total number of equations is equal to the total number of unknowns.

Example 6: Determine the reaction for the following truss.

Apply equilibrium:

\[ \sum M_A = 0; \quad (1.5 \text{ in}) B(\text{syb}) - 9.81 \text{ lb}(2 \text{ in}) - 23.5 \text{ lb}(6 \text{ in}) = 0 \quad \Rightarrow \quad B_x = 107.1 \text{ lb} \]

\[ \sum F_x = 0; \quad A_x + B_x = 0 \quad A_x + 107.1 \text{ lb} = 0 \quad \Rightarrow \quad A_x = -107.1 \text{ lb}, \quad \text{or} \quad A_x = 107.1 \text{ lb} \]

\[ \sum F_y = 0; \quad A_y - 9.81 \text{ lb} - 23.5 \text{ lb} = 0 \quad \Rightarrow \quad A_y = 33.3 \text{ lb} \]

• Two and three force members:

For a two-force member to be in equilibrium, the force system must meet three conditions: a) the two forces have the same line of action, b) equal magnitude, c) opposite sense.
For a three-force member to be in equilibrium, the force system must be concurrent or parallel.

**9) Trusses**

A truss structure is composed of a number of truss members pinned connected at their ends such that each member carries the load through axial tension or axial compression. All members must then be two-force members.

- **Internal determinacy of trusses**

Let \( j \) = number of joints, \( r \) = number of unknown reactions, and \( b \) = number of members. Then, \( 2j \) gives the number of equilibrium equations (since there are two equations per joint), and \( b + r \) gives the number of unknowns.

- If \( b + r = 2j \), the truss is classified as statically determinate internally.
- If \( b + r > 2j \), the truss is classified as statically indeterminate internally.

Finally, a truss is classified as unstable if:
- \( b + r < 2j \),
- or \( b + r \geq 2j \) if truss support reactions are concurrent or parallel.
• Simple methods of analysis for trusses

We are usually interested in solving for the internal member forces and the reactions. Two methods:

a) Method of Sections

Example 7: Find the internal force in members BC, BF, and EF by method of sections.

Draw a FBD of portion to the right of section n-n.

Apply equilibrium:

\[ \sum M_B = 0; \quad -(12\text{ft})F_{EF} - (24\text{ft})2000\text{lb} = 0 \Rightarrow F_{EF} = -4000\text{lb}, \text{ or } F_{EF} = 4000\text{lb comp} \]

\[ \sum F_y = 0; \quad F_{BF}(\sin 45^\circ) - 2000\text{lb} = 0 \Rightarrow F_{BF} = 2828\text{lb tension} \]

\[ \sum F_x = 0; \quad -F_{BC} - F_{BF}(\cos 45^\circ) - F_{EF} = 0 \]

\[ \Rightarrow -F_{BC} - (2828\text{lb})(\cos 45^\circ) - (-4000\text{lb}) = 0 \Rightarrow F_{BC} = 2000\text{lb tension} \]

b) Method of Joints

Example 8: Solve Example 7 using the method of joints.

First, draw a FBD of each joint
Example 8: cont.

Apply equilibrium conditions to each joint:

For this case, we have a total of 12 independent equilibrium equations (6 joints × 2 equations per joint = 12 equilibrium equations). We also have a total of 12 unknowns (9 internal forces and 3 reactions).

\[ \begin{align*}
\pm A_x &= 0 \implies H_A + F_{AB} + F_{AB}(1/\sqrt{2}) = 0 \quad \implies H_A = 0 \\
+ \uparrow A_y &= 0 \implies V_A - F_{AE}(1/\sqrt{2}) = 0 \quad \implies V_A = 4000\text{lb} \downarrow \\
\pm B_x &= 0 \implies F_{BC} - F_{AB} + F_{BF}(1/\sqrt{2}) = 0 \quad \implies F_{BA} = 4000\text{lb tension} \\
+ \uparrow B_y &= 0 \implies -F_{BE} - F_{BF}(1/\sqrt{2}) = 0 \quad \implies F_{BE} = 2000\text{lb comp.} \\
\pm C_x &= 0 \implies F_{CD} - F_{BC} = 0 \quad \implies F_{CB} = 2000\text{lb tension} \\
+ \uparrow C_y &= 0 \implies -F_{CF} = 0 \quad \implies F_{CF} = 0 \\
\pm D_x &= 0 \implies -F_{CD} - F_{DF}(1/\sqrt{2}) = 0 \quad \implies F_{CD} = 2000\text{lb tension} \\
+ \uparrow D_y &= 0 \implies -F_{DF}(1/\sqrt{2}) = 2000\text{lb} \quad \implies F_{DF} = 2828\text{lb comp.} \\
\pm E_x &= 0 \implies F_{EF} - F_{AE}(1/\sqrt{2}) = 0 \quad \implies F_{EA} = 5656\text{lb comp.} \\
+ \uparrow E_y &= 0 \implies V_E + F_{BE} + F_{AE}(1/\sqrt{2}) = 0 \quad \implies V_E = 6000\text{lb} \uparrow \\
\pm F_x &= 0 \implies F_{DF}(1/\sqrt{2}) - F_{BF}(1/\sqrt{2}) - F_{EF} = 0 \quad \implies F_{FE} = 4000\text{lb comp.} \\
+ \uparrow F_y &= 0 \implies F_{CF} + F_{BF}(1/\sqrt{2}) + F_{DF}(1/\sqrt{2}) = 0 \quad \implies F_{BF} = 2828\text{lb tension} \\
\end{align*} \]

Note, we obtained the reactions without applying the equations of equilibrium to the entire truss. These equations can now be applied to check the results of the analysis by the method of joints.

Global equilibrium:

\[ \begin{align*}
+ \sum M_A &= 0; \quad (12\text{ft})V_E - (36\text{ft})2000\text{lb} = 0 \implies V_E = 6000\text{lb} \uparrow \\
+ \sum F_y &= 0; \quad V_A + V_E = 0 \implies V_A = 6000\text{lb} \downarrow \\
+ \sum F_x &= 0; \quad H_A = 0 \\
\end{align*} \]

These values are the same as those obtained by the method of joints.

(10) Frames

A frame structure is composed of a number of frame members pinned (or fixed) connected at their ends. Unlike the truss, frames may contain multi-force members. The problems relating to frames usually involve determining the reactions at the ends of each member. These can only be accomplished by drawing FBD’s of all members and the entire frame.

- Solution process
  - a) Find reactions
  - b) Break structure into parts
  - c) look for two force members
  - d) find joint reactions

- Internal determinacy of frames
  If the total number equations equals the total number of unknowns, the frame is statically determinate internally. Frames can be statically indeterminate externally, but statically determinate internally.
Example 9: Determine the internal forces at each pin and the support reactions for the following frame.

![Frame Diagram]

**a) Find external reactions**

+ ∑ $M_E = 0; -(3.6\text{ft})2400\text{lb} + (4.8\text{ft})V_F = 0 \Rightarrow V_F = 1800\text{lb}$

+ $\sum F_y = 0; V_E - 2400\text{lb} + 1800\text{lb} = 0 \Rightarrow V_E = 600\text{lb}$

± $\sum F_x = 0; H_E = 0$

**b) Find internal reactions**

1. Three equations of equilibrium for each FBD, 6 unknown pin reactions (plus 3 external reactions).

   1. $\sum M_A = 0; -(2.7\text{ft})B_x = 0 \Rightarrow B_x = 0$
   2. $\sum F_x = 0; A_x - B_x = 0 \Rightarrow A_x = 0$
   3. $\sum M_B = 0; (2.4\text{ft})C_y + 2400\text{lb}(3.6\text{ft}) = 0 \Rightarrow C_y = 3600\text{lb}$
   4. $\sum F_x = 0; B_x + C_x = 0 \Rightarrow C_x = 0$
   5. $\sum F_y = 0; 1800\text{lb} - 3600\text{lb} - A_y = 0 \Rightarrow A_y = -1800\text{lb}$ wrong direction
   6. $\sum F_y = 0; -1800\text{lb} - B_y + 600\text{lb} = 0 \Rightarrow B_y = -1200\text{lb}$ wrong direction

---

10
Example 10: Determine the internal forces at each pin and the support reactions for the following frame.

Draw a FBD of each element; three equations of equilibrium for each FBD, 6 unknowns.

\[ \sum M_A = 0; \]
\[ (12\text{ft})B_y - (25\text{ft})B_x - 30\text{lb}(15\text{ft}) = 0 \]
\[ B_y - (2.083)B_x = 37.5\text{lb} \]
\[ 2 \text{ equations 2 unknowns} \]
\[ B_y - (2.083)B_x = 37.5\text{lb} \]
\[ \text{add} \frac{3.1333B_x}{-B_x - (1.05)B_x = 20\text{lb}} = -57.5\text{lb} \]
\[ \Rightarrow B_x = -18.35\text{lb} \]
\[ \sum F_y = 0; \quad A_x - 18.35\text{lb} = 0 \]
\[ \Rightarrow A_y = 0.73\text{lb} \]
\[ \sum F_x = 0; \quad A_x + 30\text{lb} - 18.35\text{lb} = 0 \]
\[ \Rightarrow A_x = 11.5\text{lb} \]

\[ \sum M_C = 0; \]
\[ (20\text{ft})B_y + (21\text{ft})B_x + 40\text{lb}(10\text{ft}) = 0 \]
\[ B_y + (1.05)B_x = -20\text{lb} \]
\[ B_y + (1.05)(-18.35\text{lb}) = -20\text{lb} \]
\[ B_y - 19.27\text{lb} = -20\text{lb} \]
\[ \Rightarrow B_y = -0.73\text{lb} \quad \text{wrong direction} \]
\[ \sum F_y = 0; \quad C_x - 40\text{lb} + 0.73\text{lb} = 0 \]
\[ \Rightarrow C_y = 39.3\text{lb} \]
\[ \sum F_x = 0; \quad C_x + 18.35\text{lb} = 0 \]
\[ \Rightarrow C_x = 18.35\text{lb} \]

Check global equilibrium
\[ \sum F_y = 0; \]
\[ 30 - 11.65 - 18.35 = 0 \]
\[ \sum F_x = 0; \]
\[ 0.73 + 37.27 - 40 = 0 \]
\[ \sum M_A = 0; \]
\[ 73.4 + 11.78 - 450 - 800 = 0 \]
(11) Internal forces and moments in multi-force members

When dealing with truss members, we found that the internal member force system consists of a single axial load with its line of action oriented along the axis of the member. For multi-force members, the internal member force system consists of two forces (one parallel to the axis of the member the other perpendicular) and a couple.

Example 11: Determine the internal forces and moment at section n-n.

\[ \begin{align*}
R_A &= \text{unknown} \\
R_B &= \text{unknown}
\end{align*} \]

### a) Find external reactions

\[ \begin{align*}
\sum M_A &= 0; \quad \text{yields } R_{By} = 220 \text{kips} \\
\sum F_y &= 0; \quad \text{yields } R_{Ay} = 320 \text{kips} \\
\sum F_x &= 0; \quad \text{yields } R_{Ax} = 0
\end{align*} \]

### b) Draw FBD of bottom portion

\[ \begin{align*}
\sum M_{nn} &= 0; \quad -M + 220 \text{kips}(5 \text{ft}) = 0 \\
\Rightarrow M &= 1100 \text{kips} \cdot \text{in} \\
\sum F_y &= 0; \quad \text{yields } F = 220 \text{kips} \\
\sum F_x &= 0; \quad \text{yields } V = 0
\end{align*} \]

### c) Check these results using the FBD of top portion

\[ \begin{align*}
\sum M_{nn} &= 0; \\
1100 \text{kips} \cdot \text{ft} - 600 \text{kips} \cdot \text{ft} - 500 \text{kips} \cdot \text{ft} &= 0 \\
\sum F_y &= 0; \\
320 \text{kips} - 100 \text{kips} - 220 \text{kips} &= 0 \\
\sum F_x &= 0; \quad 0 = 0
\end{align*} \]

(12) Cables and pulleys

Example 12: The system is in equilibrium. What is the magnitude of W?

\[ \begin{align*}
W &= 1000 \text{N} \\
\beta &= 45^\circ
\end{align*} \]

**Force polygon**

\[ \begin{align*}
1 \text{kN} & \quad 45^\circ \\
2 \text{kN} & \quad \text{W}
\end{align*} \]

**Law of cosines**

\[ W^2 = (1 \text{kN})^2 + (2 \text{kN})^2 - 2(1 \text{kN})(2 \text{kN}) \cos 45^\circ \]

\[ W = 1473 \text{N} \]

**FBD**

\[ \begin{align*}
2000 \text{N} & \quad 45^\circ \\
1000 \text{N} & \quad \text{W}
\end{align*} \]

C is the answer
Example 13: Find the tension, $T$, that must be applied to pulley $A$ to lift the 1200 N weight.

- (A) 100 N
- (B) 300 N
- (C) 400 N
- (D) 600 N

\[
\begin{align*}
\sum F &= 0; \quad 4T + 8T = 1.2kN \implies T = 100N \\
A \text{ is the answer}
\end{align*}
\]

(13) Friction

\[F_k = \mu_k N \quad \text{kinetic-friction force} \]
\[F_m = \mu_s N \quad \text{static-friction force (limiting friction)}\]

Example 14: What force $P$ must be applied to the block for it to move?

\[P_x = P_y = 0.707P \quad \mu_s = 0.2 \]
\[N = 1000 lb + 0.707 P \]
\[P_x = F = \mu_s N \implies 0.707P = 0.2(1000lb + 0.707P) \implies P = 353.6 lb \text{ needed to move the block} \]

Example 15:

\[
\begin{align*}
300lb \left( \frac{4}{5} \right) &= 240lb \\
300lb \left( \frac{3}{5} \right) &= 180lb \\
F_1 = ? \\
100 lb \\
N &= 240 lb \\
F_2 = ? \\
\mu_s &= 0.25 \\
F &= 0.25(240lb) = 60lb \\
The block will slide downhill since 100lb + 60lb < 180lb
\end{align*}
\]

(14) Centroid of lengths, areas, volumes

Associated with every line, plane area, and volume, there is a point $C$, known as the centroid, whose coordinates are

- line, $L$
  \[
  \bar{x} = \frac{Q_y}{L} \]
- plane area, $A$
  \[
  \bar{x} = \frac{Q_y}{A} \quad \bar{y} = \frac{Q_x}{A} \quad \bar{z} = \frac{Q_z}{A}
  \]
- volume, $V$
  \[
  \bar{x} = \frac{Q_y}{V} \quad \bar{y} = \frac{Q_x}{V} \quad \bar{z} = \frac{Q_z}{V}
  \]

where $Q$’s are the first moments of the quantities, $L, A, V$.

The quantities are defined in the following figure for a plane area.
Example 16: Find the centroid of the following rectangular area using integration.

- **Area of the rectangle**
  \[ A = \int_A dA = \int_0^b \int_0^h x \, dy \, dx = \frac{bh^2}{2} \]

- **First moment about x**
  \[ Q_x = \int_A y \, dA = \int_0^h \int_0^b y \, dx \, dy = \frac{bh^3}{2} \]

- **x coordinate of centroid**
  \[ \bar{x} = \frac{Q_y}{A} = \frac{b^2 h/2}{bh} = \frac{b}{2} \]

- **First moment about y**
  \[ Q_y = \int_A x \, dA = \int_0^h \int_0^b x \, dy \, dx = \frac{bh^2}{2} \]

- **y coordinate of centroid**
  \[ \bar{y} = \frac{Q_x}{A} = \frac{h^2 b/2}{bh} = \frac{h}{2} \]

Example 17: Find the centroid of the following composite area.

- **Area of the circle**
  \[ A = \frac{\pi r^2}{2} = 25.12 \text{ cm}^2 \]

- **Area of the rectangle**
  \[ A = \frac{4 \times 3 \pi}{3 \pi} = 1.6985 \text{ cm} \]

<table>
<thead>
<tr>
<th>Item</th>
<th>A(cm²)</th>
<th>(\bar{x}(\text{cm}))</th>
<th>(\bar{y}(\text{cm}))</th>
<th>(x^2 A(\text{cm}^3))</th>
<th>(y^2 A(\text{cm}^3))</th>
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<th>(y)</th>
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<td>Rect.</td>
<td>108</td>
<td>4.5</td>
<td>12</td>
<td>486</td>
<td>1296</td>
<td>(x = \frac{384}{109.9} = 3.49 \text{ cm})</td>
<td>(y = \frac{1103}{109.9} = 10.04 \text{ cm})</td>
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<td>384</td>
<td>1103</td>
</tr>
</tbody>
</table>

(15) **Center of gravity of a mass**

Associated with every body, there is a point CG, known as the center of gravity, whose coordinates are
In 2-D
\[ \bar{x} = \frac{\int x dW}{\int dW} = \frac{\sum x_i \Delta W_i}{\sum \Delta W_i} \]
\[ \bar{y} = \frac{\int y dW}{\int dW} = \frac{\sum y_i \Delta W_i}{\sum \Delta W_i} \]

Example 18: Find the center of gravity for the three objects.

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
Item & W(kg) & \(\bar{r}(cm)\) & \(rW(cm\cdot kg)\) \\
\hline
cube & 10 & 4.5 & 5 \\
rod & 5 & 3.0 & 17.5 \\
sphere & 3 & 7.3 & 21 \\
\hline
\text{sum} & 18 & & 43.5 \\
\hline
\end{tabular}
\end{center}

\[ \bar{x} = \frac{43.5}{18} = 2.42\text{cm} \]

(16) Other properties of areas

- **Moments of inertia of an area (second moments of an area)***

  The second moment about the \(x\) axis \(I_x\) and the second moment about the \(y\) axis \(I_y\) are:

  \[ I_x = \int_A y^2 dA = \sum_i \int_{A_i} y^2 dA_i, \quad \text{and} \quad I_y = \int_A x^2 dA = \sum_i \int_{A_i} x^2 dA_i \]

Example 19: Determine the moment of inertia about the centroidal \(y\) axis of a rectangular area.

- Second moment about the \(y\) axis

  \[ I_y = \int_A x^2 dA = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} x^2 dxdy = \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{x^3}{3} \bigg|_{-\frac{h}{2}}^{\frac{h}{2}} dy = \frac{b^{3}h}{12} \]

- **Parallel axis theorem**

  The moment of inertia is known about an axis, the moment of inertia about another parallel axis can be calculated using the parallel axis theorem, also known as the transfer axis theorem. This theorem is usually needed to evaluate the moment of inertia of a composite area when the centroids of the simple areas do not coincide.

  \[ I_x = I_{xc} + d_x^2 A, \quad \text{and} \quad I_y = I_{yc} + d_y^2 A. \]

- **Radius of gyration of an area**, \(r\), can be interpreted as a distance at which the entire area (as a strip) can be placed.

  \[ r = \sqrt{\frac{I}{A}} \]

- **Polar moment of inertia of an area**, \(J\), is associated with the torsional behavior of shafts:
\[ J_z = \int_A (x^2 + y^2)\,dA, \quad \text{or} \quad J_z = I_x + I_y \]

**Example 20:** Find the moments of inertia, polar moment of inertia, and radius of gyration.

<table>
<thead>
<tr>
<th>Item</th>
<th>(I_x(cm^4))</th>
<th>(I_y(cm^4))</th>
<th>(A(cm^2))</th>
<th>(x(cm))</th>
<th>(y(cm))</th>
<th>(A\bar{x}^2(cm^6))</th>
<th>(A\bar{y}^2(cm^6))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.5</td>
<td>2.0</td>
<td>6</td>
<td>3</td>
<td>4.5</td>
<td>54</td>
<td>121.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>18</td>
<td>6</td>
<td>5</td>
<td>2.5</td>
<td>150</td>
<td>37.5</td>
</tr>
<tr>
<td>(\sum)</td>
<td>5</td>
<td>20</td>
<td>12</td>
<td></td>
<td></td>
<td>204</td>
<td>159</td>
</tr>
</tbody>
</table>

\(I_x = 5.0cm^4 + 159cm^4 = 164cm^4\)
\(r_x = \sqrt{164cm^4 / 12cm^2} = 3.7cm\)
\(I_y = 20cm^4 + 204cm^4 = 224cm^4\)
\(r_y = \sqrt{224cm^4 / 12cm^2} = 4.3cm\)
\(J_o = 224cm^4 + 164cm^4 = 388cm^4\)

**Product of inertia of an area** (this is zero when either axes is an axis of symmetry)

\[ I_{xy} = \int_A xy\,dA, \quad I_{xz} = \int_A xz\,dA, \quad \text{and} \quad I_{yz} = \int_A yz\,dA. \]

**Example 21:** Find the centroid of the following composite area.

<table>
<thead>
<tr>
<th>Item</th>
<th>(A(cm^2))</th>
<th>(\bar{x}(cm))</th>
<th>(\bar{y}(cm))</th>
<th>(\bar{x}A(cm^4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5</td>
<td>-1.25</td>
<td>1.75</td>
<td>-3.28</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>1.25</td>
<td>-1.75</td>
<td>-3.28</td>
</tr>
<tr>
<td>(\sum)</td>
<td>109.9</td>
<td></td>
<td></td>
<td>-6.56</td>
</tr>
</tbody>
</table>

\(I_o = \sum \bar{x}A = -6.56cm^4\)