5.3 Derivatives and Applications

a. Derive the rules of differentiation for polynomial, trigonometric, and logarithmic functions using the formal definition of derivative

1. What does the derivative represent?

The derivative of \( f \) at \( x = c \) represents the *instantaneous rate of change* of \( f \) at \( x = c \). Graphically, the derivative represents the slope of the graph of \( y = f(x) \) at \( x = c \) (or the slope of the tangent line to \( y = f(x) \) at \( x = c \)).

2. What is the formal definition of the derivative at a point?

Finding the slope usually requires two points. (See left picture.) But to find the slope of a tangent line, you can find the slope between two points and then take the limit as those two points move closer toward each other. (See right picture.) Mathematically, the derivative of \( f \) at \( c \) is

\[
f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.
\]

There is an equivalent definition in the Sample Problems. Notice that the numerator and denominator BOTH approach zero, which means that we cannot calculate the limit just by plugging in \( h = 0 \) immediately. We have to be more thoughtful.

3. What is the formal definition of the derivative function?

Next we can define a function \( f'(x) \) whose value at \( x = c \) is the slope of \( f(x) \) at \( x = c \). That is,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

The derivative function can also be written as \( \frac{df}{dx} \).

4. What are some properties of derivatives?

One can show from the definition that the derivative of \( kf(x) \) is \( kf'(x) \) if \( k \) is a constant. One can also show:

- the derivative of \( f(x) \pm g(x) \) is \( f'(x) \pm g'(x) \);
• the derivative of \( f(x)g(x) \) is \( f'(x)g(x) + f(x)g'(x) \) [Product Rule];

• the derivative of \( \frac{f(x)}{g(x)} \) is \( \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \) [Quotient Rule]; and

• the derivative of \( f(g(x)) \) is \( f'(g(x))g'(x) \) [Chain Rule].

5. How do you use the definition to differentiate polynomial and trigonometric functions?

Probably the most straightforward derivative examples are of the powers of \( x \). As an example, let’s find the derivative of \( f(x) = x^2 \). We’ll use the first definition.

\[
f'(x) = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} (2x + h) = 2x.
\]

Next we’ll consider a trigonometric example. Let’s find the derivative of \( \sin x \) at \( x = 0 \).

\[
\lim_{h \to 0} \frac{\sin(0 + h) - \sin 0}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.
\]

This means that the slope of the graph \( y = \sin x \) is exactly 1 at \( x = 0 \) (WHEN \( x \) IS GIVEN IN RADIANS). [For a geometric reason why this limit is exactly 1, one can look online.]

6. How do you use the definition to differentiate logarithmic functions?

Here’s our plan:

(a) Look at the derivative of an exponential function.

(b) Figure out how the derivative of an inverse function relates to the derivative of the original function.

(c) Find the derivative of a logarithmic function.

Here we go:

(a) Let’s look at the exponential function \( f(x) = e^x \). Its derivative is

\[
f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = e^x \left( \lim_{h \to 0} \frac{e^h - 1}{h} \right) = e^x.
\]

Yes, as \( h \to 0 \), the limit of \( \frac{e^h - 1}{h} \) is 1. This is related to the fact that \( e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \).

(b) Next, consider \( f^{-1}(x) \). Since the graph of \( y = f^{-1}(x) \) is the same as the graph of \( y = f(x) \) only reflected in the line \( y = x \), the slopes at corresponding points are reciprocals of each other (because \( \frac{dy}{dx} \) becomes \( \frac{dx}{dy} \)).

(c) So, the derivative of \( y = \ln x \) at \( x = 2 \), say, is the reciprocal of the derivative of \( y = e^x \) at \( y = 2 \) (and \( x = \ln 2 \)). Since the derivative of \( f(x) = e^x \) is \( f'(x) = e^x \), its slope at \( \ln 2 \) is 2. So the derivative of \( \ln x \) at 2 is \( \frac{1}{2} \). Similarly, the slope of \( y = \ln x \) at \( x = c > 0 \) is \( \frac{1}{c} \). Therefore, the derivative function of \( y = \ln x \) is \( y' = \frac{1}{x} \).
7. What is implicit differentiation? . . . logarithmic differentiation?

Implicit differentiation can be used to find \( \frac{dy}{dx} \) when \( y \) is an implicit function of \( x \). The idea is to take the derivative as usual, but remember that \( y \) is a function of \( x \), and so it requires the Chain Rule to be properly differentiated. For example, to find the slope of the circle \( x^2 + y^2 = 5 \) at the point \((-1, -2)\), we can use implicit differentiation.

\[
x^2 + y^2 = 5 \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.
\]

So, at \((-1, -2)\), we have \( \frac{dy}{dx} = -\frac{1}{2} \).

Sometimes, one has to take a logarithm of both sides first, which leads some people to use the term “logarithmic differentiation.” See Sample Problems for an example.

8. Sample Problems

(a) Find a formula for the derivative of \( y = 3x^2 + x \) using the formal definition.

(b) Show that the derivative of \( x^n \) is \( nx^{n-1} \).

(c) Using the definition of the derivative, find the derivative of \( \frac{1}{x} \) and of \( \frac{1}{x^2} \).

(d) Use the Product Rule to find the derivative of \( x^2 = x \cdot x \).

(e) Use the Quotient Rule to find the derivative of \( \tan x = \frac{\sin x}{\cos x} \).

(f) Use the Chain Rule to find the derivative of \( \sin(3x^2) \) and \( \ln(1 + x^2) \).

(g) Another way to define the derivative of \( f \) at \( x = c \) is

\[
f'(c) = \lim_{b \to c} \frac{f(b) - f(c)}{b - c}.
\]

Explain why this definition is equivalent to the one given above. Draw a picture showing your reasoning.

(h) Using the alternate definition of the derivative, find derivatives of \( x^2 \) and \( \frac{1}{x} \).

(i) Using the definition of the derivative, find the derivative of \( y = \cos x \). You may use the fact that \( \lim_{h \to 0} \frac{\sin h}{h} = 1 \) and \( \lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \).

(j) Find the derivative of \( y = x^x \). [Hint: Use logarithmic differentiation by taking the logarithm of both sides and then implicitly differentiating the resulting equation.]

9. Answers to Sample Problems

(a) Find a formula for the derivative of \( y = 3x^2 + x \) using the formal definition.

\[
f'(x) = \lim_{h \to 0} \frac{3(x + h)^2 + (x + h) - (3x^2 + x)}{h}
= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + x + h - 3x^2 - x}{h}
= \lim_{h \to 0} \frac{6xh + 3h^2 + h}{h} = \lim_{h \to 0} (6x + 3h + 1) = 6x + 1.
\]