Math 53
Final Exam Review with Answers

None of these problems will appear on the final exam. However, if you can complete these problems using only the integral tables in your textbook, then you have a good chance of performing well on the final.

1. Find the exact value of the following definite integrals.
   (a) $\int_{0}^{1} xe^{x+1} \, dx = e$
   
   (b) $\int_{0}^{\pi} \sin^{2}\left(\frac{x}{3}\right) \, dx = \frac{\pi}{2} - \frac{3\sqrt{3}}{8}$

   (c) $\int_{1}^{e} (1 + \ln x) \sin^{3}(x \ln x) \, dx = -\frac{1}{3}(\cos e)(\sin^{2} e + 2) + \frac{2}{3} = -\frac{1}{3}(\cos e)(3 - \cos^{2} e) + \frac{2}{3}$

   (d) $\int_{0}^{\pi} (e^{2x} + e^{2x} \cos x) \, dx = \frac{1}{10}(e^{2\pi} - 9)$

   (e) $\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^{2} x} \, dx = \frac{\pi}{4}$

2. Integrate. Assume that $a$ is a constant.
   (a) $\int z \ln(az) \, dz = \frac{1}{2} z^{2} \ln(az) - \frac{1}{4} z^{2} + C$

   (b) $\int \frac{dw}{\sqrt{a + w}} = 2\sqrt{a + w} + C$

   (c) $\int (\sin^{5}(ax) \cos(ax) + \cos^{5}(ax) \sin(ax)) \, dx = \frac{1}{6a} [\sin^{6}(ax) - \cos^{6}(ax)] + C$

   (d) $\int \frac{1 + e^{at}}{at + e^{at}} \, dt = \frac{1}{a} \ln |at + e^{at}| + C$

   (e) $\int \frac{dx}{x^{2} + 6x + 10} = \arctan(x + 3) + C$
3. Find the radius of convergence and the interval of convergence for each series.

   (a) \( \sum_{k=1}^{\infty} \frac{x^k}{k} \); \( R = 1 \); converges on \([-1, 1)\)

   (b) \( \sum_{k=1}^{\infty} \frac{x^k}{2^k} \); \( R = 2 \); converges on \((-2, 2)\)

   (c) \( \sum_{k=1}^{\infty} \frac{x^k}{k!} \); \( R \) is infinite; converges for all \( x \)

4. Mathematicians have shown that

   \[ \int_0^\pi \ln(\sin x) \, dx = -\frac{\pi}{2} \ln 2. \]

   (a) Why is this an improper integral? This is an improper integral of Type II because the integrand is not defined at zero. I.e., \( \ln(\sin(0)) \) is undefined.

   (b) Why does this integral have a negative value? The integral is negative because the integrand is negative. Since \( \sin x \leq 1 \), \( \ln(\sin x) \leq 0 \).

   (c) Using this formula, find \( \int_0^{2\pi} \ln(\sin x) \, dx \). ANS: \(-2\pi \ln 2\)

5. Find an exact value for \( \int_0^\infty \frac{x}{e^{3x} + 1} \, dx \) if you know that \( \int_0^\infty \frac{x}{e^x + 1} \, dx = \frac{\pi^2}{12} \). ANS: \( \frac{\pi^2}{108} \)

6. Consider the region bounded between \( y = x^2 \) and \( y = \sqrt{x} \).

   (a) Find a good approximation for the perimeter of this region. \( \frac{1}{2} [2\sqrt{5} + \ln(2 + \sqrt{5})] \approx 2.958 \)

   (b) Find the exact area of this region. \( \frac{1}{3} \)

   (c) Find the volume of the solid obtained by rotating this region about the \( x \)-axis. \( \frac{3\pi}{10} \)

   (d) Find the volume of the solid obtained by rotating this region about the line \( x = -2 \). \( \frac{49\pi}{30} \)

   (e) Find the volume of the solid having this region as its base and having slices perpendicular to the \( x \)-axis that are semicircles. \( \frac{9\pi}{560} \)

7. For each integral, determine whether it converges or diverges. If it converges, find what it converges to. Answer exactly if possible.
(a) \[ \int_{3(\theta - \pi)}^{4} d\theta \] diverges

(b) \[ \int_{3\sqrt{\theta - \pi}}^{4} d\theta \] converges to \( 2\sqrt{4 - \pi} \)

(c) \[ \int_{0}^{\infty} e^{-qw} \, dw \], where \( q > 0 \); converges to \( \frac{1}{q} \)

(d) \[ \int_{1}^{\infty} \frac{x^2}{x^3 + 1} \, dx \] diverges

8. The circumference \( C \) of a tree at various heights is given below. Find a good upper and lower bound for the volume of the tree. What assumptions are you making? Notice the units.

<table>
<thead>
<tr>
<th>height, in ft</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C ), in inches</td>
<td>61</td>
<td>45</td>
<td>33</td>
<td>21</td>
<td>12</td>
<td>5</td>
</tr>
</tbody>
</table>

I am assuming that each slice is circular, that each segment is a cylinder, and that the radius of the tree is decreasing. Under these assumptions, an overestimate for the volume of the tree is 12,660 in\(^3\) and an underestimate is 6210 in\(^3\).

9. Values of \( f'(x) \) are given below. Complete the table.

\[
\begin{array}{c|cccc}
 x & 0 & 2 & 4 & 6 \\
 \hline
 f'(x) & -1 & 0 & 3 & 2 \\
 f''(x) & \frac{7}{2} & 1 & \frac{1}{2} & -\frac{7}{2} \\
 f(x) & 1 & 0 & 3 & 8 \\
\end{array}
\]

To fill in the table, I used data from the right and left sides to find \( f''(2) \) and \( f''(4) \). To find \( f(2) \), I assumed that \( f'(x) \) was a line between \((0, 1)\) and \((2, 0)\) and then used the Fundamental Theorem of Calculus.

10. Approximate \( \int_{0}^{1} \sqrt{x - x^3} \, dx \) to two digits of accuracy. Explain, using the shape of the graph of \( \sqrt{x - x^3} \), why you are absolutely certain that your answer is correct.

I graphed the integrand to find that it was concave down on the interval \( 0 \leq x \leq 1 \). Hence, the Trapezoidal Rule will give an underestimate while the Midpoint Rule will give an overestimate. Using ALLSUMS on my calculator with 40 subdivisions, I found \( \text{TRAP}(40) \approx 0.47727 \) and \( \text{MID}(40) \approx 0.4798 \). Therefore, 0.478 is an approximation to the integral that has two digits of accuracy; i.e., the error is less than 0.005. [0.48 will also work. 0.47 is incorrect.]

11. Repeat the previous problem for \( \int_{0}^{1} e^{-x^2/2} \, dx \).

This integrand is decreasing on the interval \( 0 \leq x \leq 1 \). Hence, left hand sums will provide an overestimate while right hand sums will be under. So I need to choose enough subdivisions so that the difference between Left and Right Hand Sums is less than 0.01. Then by choosing their average, I know my answer will be within 0.005 of the correct one. By trial and error, I found that 40 subdivisions gives \( \text{LEFT} \approx 0.8605 \) and \( \text{RIGHT} \approx 0.8507 \). So, 0.8556 is accurate to two decimal places.
12. Find the parabola of best fit to the function \( y = \sqrt{x^3 + 1} \) .
   
   (a) ... near \( x = 0 \). \( P_2(x) = 1 \).
   
   (b) ... near \( x = 2 \). \( P_2(x) = 3 + 2(x - 2) + \frac{1}{3}(x - 2)^2 \).

13. Find the first three nonzero terms of the Taylor series for \( 2 \sin x \cos x \) near \( x = 0 \). Explain how this supports the trigonometric identity \( \sin 2x = 2 \sin x \cos x \).
   
   You need to find all the terms of degree 5 or less.

\[
2 \sin x \cos x = 2 \left( x - \frac{x^3}{6} + \frac{x^5}{120} \right) \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} \right) \\
= 2 \left[ x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^5}{6} + \frac{x^5}{12} + \frac{x^5}{120} \right] \\
= \text{some algebra...} \\
= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} \\
= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} \\
= (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!},
\]

which agrees with the first three terms of \( \sin(2x) \).

14. Find a polynomial function \( f(x) \) with \( f(3) = f'(3) = f''(3) = f'''(3) = f''''(3) = 30 \).

\( f(x) = 30 + 30(x - 3) + 15(x - 3)^2 + 5(x - 3)^3 + \frac{5}{4}(x - 3)^4 \).

15. Consider \( F(x) = \int_0^x e^{-t^2} dt \).

   (a) Find \( F'(x) \). \( F'(x) = e^{-x^2}[= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...] \)

   (b) Find the Taylor series of \( F(x) \). [Hint: find the series for \( F'(x) \) first.]

\[ F(x) = x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + ... \]

16. Explain why \( \int_1^\infty x^{-2} \, dx \) converges while \( \int_1^\infty x^{-1/2} \, dx \) diverges. ANSWERS MAY VARY. (Explain in terms of the limit of a sequence of definite, proper integrals.)

17. A physicist determines that the amount of work needed to perform a certain task is \( \int_3^\infty \frac{dx}{x \ln x} \). Is this amount finite or infinite? Explain.

If the integral were \( \int_3^B \frac{dx}{x \ln x} \), then it would equal \( \ln(\ln B) - \ln(\ln 3) \). Since this quantity approaches infinity (grows without bound) as \( B \) approaches infinity, the amount of work needed to perform the task is infinite.
18. A water tank in the shape of an inverted cone is buried so that the top of the tank (the base of the cone) is 20 feet below ground. The bottom of the tank lies 50 feet below ground. If the radius of the cone at the top is 10 feet, find the total amount of work required to pump all the water up to ground level. (Water weighs 62.4 pounds per cubic foot.) ANS: \((62.4\pi)(27500) \approx 5.391 \times 10^6 \text{ ft-lbs}\)

19. A banner in the shape of an isosceles trapezoid hangs from a building. The larger base, 10m across, is at the top of the building, while the smaller base, 5m across, is located 7m below the roofline. Calculate the work required to haul the banner up to the roof, assuming that the banner has a density of 1.4 kg per square meter. ANS: \((9.8)\frac{686}{3} \approx 2241 \text{ Joules}\)

20. Johnny sees a rope hanging down from the roof of a building of height \(H\). The tip of the rope barely touches the ground. After testing it, he climbs the rope and then hauls the rope up to the roof. Assuming the rope has a total weight of \(R\) and Johnny has a weight of \(J\), calculate how much work Johnny did in hauling himself and the rope up to the roof. ANS: \(JH + \frac{1}{2}RH\)

21. Find the work you must do against gravity to construct a regular tetrahedron of edge length 1m out of copper. The density of copper is 8.23 g/cm\(^3\). ANS: \(\frac{\sqrt{3}}{72}(8230)(9.8) \approx 1940 \text{ Joules}\) [Hint: the height of the tetrahedron (triangular pyramid) is \(\sqrt{\frac{2}{3}}\) meters.]