MATH 057 Applied Differential Equations I—Section 01

In-Class Part of Midterm Exam 1 (70%), September 16, 2009.

Instructions: Do all problems and be sure to show all work. You can use your calculators
to do simple computations, and to check integrals and graphs that you perform and
justify by hand. Be sure that your work makes this clear. You will not receive credit
unless you justify your steps.

If you make a graph be sure to label the axes, if you write down an equation be sure
every variable is defined (state units, etc.). Your work should be clear and neat.

Name: ____________________________

ID Number: ________________________

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1. (Total: 12 points) Consider the differential equation

\[
\frac{dy}{dt} = y^2 - 2(t + 1)y + (t + 1)^2
\]

a. (6 points) Show that the functions \( y_1(t) = t \) and \( y_2(t) = t + 2 \) are solutions of this differential equation.

b. (6 points) Suppose that the function \( y_3(t) \) is the solution to the above differential equation that goes through the point \((1, 2)\). Using the Existence and Uniqueness Theorem, what can you say about \( y_3(2000) \)? Include a one sentence justification and a graph for your answer.
2. (Total: 12 points) Find the general solution (9 points) of the following differential equation.

\[
\frac{dy}{dt} = \frac{(y^2 + 1)t \sin(t)}{y}
\]

Then solve the initial value problem (3 points) with \( y(0) = y_0 \) where \( y_0 > 0 \).
3. (Total: 12 points) Consider the following initial value problem (IVP)

\[ \frac{dy}{dt} = \sin(t^2) \]

\[ y(0) = 0.5 \]

Use Euler’s Method to compute the solution \( y(t) \) of the IVP for \( 0 \leq t \leq 0.75 \) with step size \( \Delta t = 0.25 \). Present the formulas that you need to use (3 points), make a table (5 points) with the computed values and graph (2 points). What would you say is the solution at \( t = 0.4 \)? (2 points)
4. (Total: 10 points) Systems a and b below are models of the populations of pairs of species. One of the systems correspond to species that compete for the same resource of food. The other system correspond to species that cooperate to get the same resource of food. Which one is which? (5 points) and why? (5 points).

a. 
\[
\begin{align*}
\frac{dx}{dt} &= \alpha x - \alpha \frac{x^2}{N} + \beta xy \\
\frac{dy}{dt} &= \gamma y + \delta xy
\end{align*}
\]

b. 
\[
\begin{align*}
\frac{dx}{dt} &= -\gamma x - \delta xy \\
\frac{dy}{dt} &= \alpha y - \beta xy
\end{align*}
\]
5. (Total: 12 points) Consider the following 8 first-order equations:

1. \( \frac{dy}{dt} = -y^4 \),  
2. \( \frac{dy}{dt} = e^y \),  
3. \( \frac{dy}{dt} = (1 - t)y \),  
4. \( \frac{dy}{dt} = y \sin(t) \)

5. \( \frac{dy}{dt} = yt \),  
6. \( \frac{dy}{dt} = -t^3 \),  
7. \( \frac{dy}{dt} = t \sin(y) \),  
8. \( \frac{dy}{dt} = ye^y \)

Four of the associated slope fields are shown on the next page. Pair the slope fields with their associated equations. Complete the sentences below and provide a brief justification for your choice. You will not receive any credit unless you justify your selection.

a. The equation for slope field A is ...... My reason for choosing this answer is:

b. The equation for slope field B is ...... My reason for choosing this answer is:

c. The equation for slope field C is ...... My reason for choosing this answer is:

d. The equation for slope field D is ...... My reason for choosing this answer is:
5. (continued) **Answer this question on the previous page.** The equations are provided here for your convenience:

1. \[ \frac{dy}{dt} = -y^4, \]
2. \[ \frac{dy}{dt} = e^y, \]
3. \[ \frac{dy}{dt} = (1 - t)y, \]
4. \[ \frac{dy}{dt} = y \sin(t) \]
5. \[ \frac{dy}{dt} = yt, \]
6. \[ \frac{dy}{dt} = -t^3, \]
7. \[ \frac{dy}{dt} = t \sin(y), \]
8. \[ \frac{dy}{dt} = ye^y \]
6. (Total: 12 points) A tank contains 300 gallons of pure water. We add water to the tank at a rate of 2 gallons per minute containing 1/2 pound of minerals per gallon of water. At the same time, we use the mixture (water and minerals) to water the lawn at a rate of 2 gallons per minute.

a. (5 points) Make a model for the amount of minerals in the tank as a function of time (write an IVP).

b. (5 points) Solve the IVP to determine the amount of minerals in the tank after 10 minutes.

c. (2 points) Make a qualitative analysis of the problem to estimate the amount of minerals in the tank after a long long time.