Math 057 Applied Differential Equations I.
Review of Techniques of Integration.
Strategy.
Examples.

1. With a table of derivatives.
\[ \int \sin(x) \, dx = -\cos(x) + c. \]

2. \( u \)-substitution.
\[ \int \sin(3x) \, dx = -\frac{1}{3} \cos(3x) + c, \]
using \( u = 3x \).

3. Integration by parts.
\[ \int x \sin(x) \, dx = -x \cos(x) + \sin(x) + c, \]
using
\[ \int u \, dv = uv - \int v \, du, \]
with \( u = x \) and \( dv = \sin(x) \, dx \).

4. Trigonometric integrals:
4a. Products of sines and cosines.
\[ \int \sin^m(x) \cos^n(x) \, dx. \]
If \( m \) or \( n \) are odd, use \( \sin^2(x) + \cos^2(x) = 1 \) and then \( u \)-substitution with \( u = \sin(x) \) if \( m \) is odd and, \( u = \cos(x) \) if \( n \) is odd. If \( m \) and \( n \) are both even use
\[ \sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \]
\[ \cos^2(x) = \frac{1}{2}(1 + \cos(2x)). \]

4b. Products of tangents and secants.
\[ \int \tan^m(x) \sec^n(x) \, dx. \]
Use \( \sec^2(x) = 1 + \tan^2(x) \) and then \( u \)-substitution with \( u = \tan(x) \) if \( n \) is even and \( u = \sec(x) \) if \( m \) is odd.
5. **Trigonometric Substitution.** This case involve integrals that contain expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$. If $u$-substitution is not an option we can use one of these trigonometric substitutions:

1. $\sqrt{a^2 - x^2}$ substitute $x = a \sin(\theta)$. Need $\cos^2(\theta) + \sin^2(\theta) = 1$.
2. $\sqrt{a^2 + x^2}$ substitute $x = a \tan(\theta)$. Need $\sec^2(\theta) - \tan^2(\theta) = 1$.
3. $\sqrt{x^2 - a^2}$ substitute $x = a \sec(\theta)$. Need $\sec^2(\theta) - \tan^2(\theta) = 1$.

Once the integral is solved in the new variable $\theta$ we need to rewrite the answer in the original variable $x$. The answers to these problems usually involve $\sec(\theta)$, $\cot(\theta)$ and other trigonometric expressions. For this task it is useful to draw a right triangle in the following way:

![Figure 1: Trigonometric substitution. The three different right triangles that help you come back to the variable x after solving the problem in the variable theta.](image)

**Exercise:** Label the different triangles in the above picture as cases 1, 2 or 3 depending on the trigonometric substitution that correspond to each one of them. Then, write expressions for $\theta$, $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\sec(\theta)$ and $\cot(\theta)$ for each case.
**Example:** Solve the following integral:

\[ \int \frac{x^2}{\sqrt{9 - x^2}} \, dx. \]

We let \( x = 3 \sin(\theta) \) and then \( dx = 3 \cos(\theta) \, d\theta \) and \( \sqrt{9 - x^2} = 3 \cos(\theta) \). Then

\[
\int \frac{9 \sin^2(\theta)}{3 \cos(\theta)} \, 3 \cos(\theta) \, d\theta = 9 \int \sin^2(\theta) \, d\theta = 9 \int \frac{1}{2} (1 - \cos(2\theta)) \, d\theta \\
= \frac{9}{2} (\theta - \frac{1}{2} \sin(2\theta)) + c = \frac{9}{2} (\theta - \sin(\theta) \cos(\theta)) + c.
\]

Finally, we write

\[
\int \frac{x^2}{\sqrt{9 - x^2}} \, dx = \frac{9}{2} \left( \sin^{-1} \left( \frac{x}{3} \right) - \left( \frac{x \sqrt{9 - x^2}}{9} \right) \right) + c.
\]

**6. Partial Fractions Decomposition.** This case involve integrals of rational functions, that is, functions of the form

\[ R(x) = \frac{P(x)}{Q(x)}, \]

where \( P(x) \) and \( Q(x) \) are polynomials.

If \( \deg P(x) \geq \deg Q(x) \) the rational function is improper. In this case we first use *long division* to transform the improper function into a proper one. Once the rational function \( R(x) \) is proper we proceed according to the following rules:

1. Factor the denominator of the rational function into linear factors (i.e., factors of the form \( ax + b \)) and irreducible quadratic factors (i.e., factors of the form \( ax^2 + bx + c \)).

2. You may also need to complete squares on the irreducible quadratic factors and use \( u \)-substitution to put them in the form \( x^2 + d \).

3. Write the rational function \( R(x) \) in *partial fractions*, i.e., write the rational function as a sum of terms with denominators corresponding to the factors of the denominator of \( Q(x) \). Remember that the numerators of the partial fractions are constants when the factor is linear and linear when the factor is quadratic. If a factor is repeated \( n \) times then you need to write down \( n \) terms with denominators to powers from 1 to \( n \).

4. Finally, integrate each partial fraction and add the results to get the integral of \( R(x) \).
**Example:** Find the antiderivative of

\[ \frac{1}{x^3(2 + x)(x^2 + 4)^2}. \]

The rational function is proper and the denominator is already factored so we now set up an equation to write it in partial fractions.

\[ \frac{1}{x^3(2 + x)(x^2 + 4)^2} = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{b}{2 + x} + \frac{c_1 x + d_1}{x^2 + 4} + \frac{c_2 x + d_2}{(x^2 + 4)^2}. \]

The next step is to find the constants \( a_i, b, c_i \) and \( d_i \) so that the two expressions are equal.

**Exercise:** Find the constants \( a_i, b, c_i \) and \( d_i \) so that the above two expressions are equal. Then, integrate each term and add the results to get the antiderivative of

\[ \frac{1}{x^3(2 + x)(x^2 + 4)^2}. \]

The last step becomes easier when one remembers the following integrals:

\[ \int \frac{1}{x - a} \, dx = \ln |x - a| + c. \]

\[ \int \frac{1}{ax - 1} \, dx = \frac{1}{a} \ln |ax - 1|. \]

\[ \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + c. \]

\[ \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + c. \]

**Trick or Treat? A Halloween Strategy for Integration.**

It is October 31 and you are walking around campus near the library. Suddenly, someone in a Halloween costume (that resembles your favorite math professor) hands in to you a piece of paper with a given integral of the form:

\[ \int f(x) \, dx. \]

This kind of ghost, all covered in spider webs, surrounded by bats and smelling like a burned squirrel tells you that you can get rid of him by solving the given integral. He
also says that if you don’t solve it, you will have horrible nightmares and you won’t be able to sleep for the rest of your life.

Before calling 911, your academic advisor or even google the integral, you may just give it a try, who knows? you may be able to get rid of the monster in a couple of minutes.

But given the variety of functions \( f(x) \) that this creepy thing can put as an integrand, what would be your strategy to attack the problem?

In this handout we describe some basic ideas that may help you attack the problem in an organized way. If you master this strategy and the techniques involved in it, even the scary ghost will run away from you in this challenge.

1. First of all, you must have in your mind a basic table of integrals that you have already worked out. You don’t want to spend time solving integrals you have already solved. You have to be able to pull out these integrals like you can pull out your parents and friends birthdays. For example, the following integrals should be in your list:

\[
\int \tan(x) \, dx = \ln |\sec(x)| + c
\]

\[
\int \sec(x) \, dx = \ln |\sec(x) + \tan(x)| + c
\]

\[
\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + c
\]

\[
\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + c
\]

2. If possible, simplify the integrand using algebraic manipulation, factoring out or expanding polynomials, using trigonometric identities, etc.

Notice that the first two steps mean basically: look carefully at the integral before you start trying to solve it because you may already know the answer or, with a little tweak, you could find it in your mind list.

3. Search for an apparent substitution. Remember that some scary looking integrals become tame after a reasonable \( u – \) substitution.

4. Recognize the form of the integral:

a. Trigonometric Functions. For example, products of powers of sines and cosines as well as products of powers of secants and tangents. You may need some trigonometric identities and some \( u – \) substitutions, too.
b. Rational Functions. You may need to use partial fractions in all of its flavors.

c. Integration by Parts. This is a reasonable idea when the integrand is a product of a polynomial and a transcendental function (e.g. the product of $x^2$ and $\sin(x)$, the product of $u$ and $e^u$, etc.). Also, the integrals of some inverse functions like $\tan^{-1}(x)$ or $\sin^{-1}(x)$ lie in this category. Remember that we even solved the integral of $\ln(x)$ in this way.

d. Radicals. There are two kinds:

1. The ones that surrender to trigonometric substitutions, i.e., integrands containing $\sqrt{x^2 - a^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$; and,

2. The ones containing $\sqrt{ax + b}$ or, more generally, something like $\sqrt[g(x)]{x}$, where $n$ is some integer and $g(x)$ a given function.

In the first case, you need to transform the given integral into a trigonometric integral by making a substitution of the form $x = a \sec(\theta)$, $x = a \sin(\theta)$ or $x = a \tan(\theta)$. Then, you are back to case [a.] Trigonometric Integrals. (and don’t forget to draw the right triangle!)

In the second case you just make the substitution $u = \sqrt{ax + b}$ (or, $u = \sqrt[g(x)]{x}$) and hope for the best!

5. Don’t give up, try again. Think about the following questions:

- Have you tried all possible $u$–substitutions?
- Have you tried all possible integration by parts?

You may have to use several methods: manipulation of the integrand, substitution, integration by parts (may be a couple of times) and then, some new clever substitution, etc. In the process of trying things you may be able to relate your problem to one that you have already solved or get a hint on what you really have to do.

Finally, if you want to be good at solving integrals before this coming Halloween, you should solve as many integrals as you can and take advantage of the 1st Midterm Exam on September 16th to assess yourself.

By the way, I almost forgot! Next semester or in a couple of years you will be taking a more advanced course in engineering, chemistry, pharmacy, mathematics, computer science, biology, etc. and you will need to solve integrals, integrals that will be very difficult to solve, much more difficult than the ones this monster can give you.

If you take the time now to learn this beautiful subject you will have a good time in those and other courses by learning new material, material that is more advanced and more closely related to your interests. Moreover, you will be able to sleep at night.

Sebastian