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THE COMPLEX DYNAMICS OF SINGULARLY PERTURBED RATIONAL MAPS

by

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To Maria Ines and Bianca
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ABSTRACT

The dynamics of singularly perturbed complex rational maps is explored. These rational maps are of the form \( f_\lambda(z) = z^n + \lambda / ((z - a)^{d_a} (z - b)^{d_b}) \) where \( n, d_a \) and \( d_b \) are integers such that \( n \geq 2, d_a, d_b \geq 1 \) and \( a, b \) and \( \lambda \in \mathbb{C} \) such that \( |a|, |b| \neq 1 \) and \( |\lambda| \) is sufficiently small. The topological characteristics of the Julia and Fatou sets of these maps are studied. The dynamics of these maps on their Julia sets are also described and modeled using symbolic dynamics. Despite the large number of possibilities we show that in most cases the Julia set of \( f_\lambda \) consists of a countable number of disjoint simple closed curves and uncountably many point components that accumulate on each of these curves. The main differences appear in the topological structure of the Fatou set of the map for different positions and orders of the poles \( a \) and \( b \). We show that the Fatou set may consist of the disjoint union of one infinitely connected component and countably many disks, every component of the Fatou set is infinitely connected, or some components of the Fatou set are infinitely connected, some are disks and some are annuli.
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Chapter 1

Introduction

In the last few years a number of papers have appeared that deal with the dynamics of functions obtained by perturbing the complex function $z^n$ by adding a pole at the origin [DLU05, DHL$^+$06, BDL$^+$05, DRS07]. These rational functions are of the form $f_\lambda(z) = z^n + \lambda/z^d$ where $n \geq 2$ and $d \geq 1$ are integers and $\lambda$ is a complex parameter. When $|\lambda| << 1$ we consider this function as a singular perturbation of $z^n$. The reason for this terminology is that, when $\lambda = 0$, the degree of the map is $n$ and the dynamical behavior is well understood. When $\lambda \neq 0$, however, the degree jumps to $n + d$ and the dynamical behavior changes significantly.

In this dissertation we study a more general class of functions for which the perturbation is given by adding a rational function of degree $d = d_a + d_b$ with two poles: $a$ of order $d_a$ and $b$ of order $d_b$ in the following way

$$f_\lambda(z) = z^n + \frac{\lambda}{(z - a)^{d_a} (z - b)^{d_b}}$$

where $n \geq 2$, $d \geq 1$ are integers, the poles $a$ and $b$ are not simultaneously zero and $\lambda$ is a sufficiently small complex parameter.

We are interested in describing the dynamics on the Julia set of $f_\lambda$, i.e., the set
of points at which the family of iterates of $f_\lambda$ is not a normal family in the sense of Montel. Equivalently, the Julia set is the closure of the set of repelling periodic points of $f_\lambda$ as well as the boundary of the set of points whose orbits escape to $\infty$. We denote the Julia set by $J = J(f_\lambda)$. The complement of the Julia set is called the Fatou set.

When $\lambda = 0$, $\infty$ and 0 are superattracting fixed points and the Julia set of $f_\lambda$ is the unit circle. When we add the perturbation by setting $\lambda \neq 0$ but very small, several aspects of the dynamics remain the same, but others change dramatically. For example, when $\lambda \neq 0$, the point at $\infty$ is still a superattracting fixed point and there is an immediate basin of attraction of $\infty$ that we call $B = B_\lambda$. The function $f_\lambda$ takes $B$ to 1 onto itself. If the pole $a$ is such that $|a| < 1$, then the preimage of $B$ that contains the pole $a$ is called a trap door and we denote this set by $T_a = T_{\lambda,a}$. Similarly, if $|b| < 1$ the trap door at $b$ is denoted by $T_b$. Note that $T_a$ and $T_b$ are disjoint from $B$ since $a$ and $b$ lie inside $\partial B$. Every point that escapes to infinity and does not lie in $B$ must do so by passing through one of the trap doors $T_a$ or $T_b$. Since the degree of $f_\lambda$ changes from $n$ to $n + d$, 2$d$ additional critical points are created. The orbits of the critical points are of fundamental importance in characterizing the Julia set of $f_\lambda$.

We first review some well known results about the case when $a = b = 0$. Notice that in this case, for sufficiently small $\lambda$ the trap door, which we denote by $T$, surrounds the origin. Let $\omega^{n+d} = 1$. Then the dynamics of $f_\lambda$ is symmetric under $z \mapsto \omega z$ since, in this case, $f_\lambda(\omega z) = \omega^n f_\lambda(z)$. Infinity and 0 are critical points of order $n - 1$ and $d - 1$, respectively. The orbits of these critical points are completely determined. One checks easily that the free critical points of $f_\lambda$ are $c_\lambda = (\lambda d/n)^{1/(n+d)}$ and the critical values are $v_\lambda = c_\lambda^n (1 + n/d)$. Since the free critical points are arranged symmetrically about the origin, their orbits also behave symmetrically and so there
is essentially only one free critical orbit. In this case we have the so-called *Escape Trichotomy* [DLU05]. This result describes the three possible types of Julia sets that arise when the critical orbits escape.

**Theorem.** (*The Escape Trichotomy*). Let $a = b = 0$ with $n \geq 2$ and $d \geq 1$ integers and $\lambda$ a complex parameter. Suppose that the free critical orbits tend to $\infty$.

1. If the critical values lie in $B$, then $J$ is a Cantor set;

2. If the critical values lie in $T$, then $J$ is a Cantor set of simple closed curves (quasi-circles);

3. If the critical values do not lie in $B$ or $T$, then $J$ is a connected set called a Sierpinski curve.

We remark that case 2 of the Escape Trichotomy is due to McMullen [McM88] and it occurs only when $1/n + 1/d < 1$. The set of parameter values $\lambda$ for which the critical values lie in $T$ is a punctured open disk in the parameter plane that surrounds the origin and is bounded by a simple closed curve. This open set is called the *McMullen domain* [Dev05, DM05].

The case $n = d = 2$ is very different. In this case there are uncountably many maps whose dynamics are different from one another in any neighborhood of the origin in parameter $\lambda$-plane (see [DHL+06]). Indeed, there are infinitely many open sets in any neighborhood of $\lambda = 0$ in which the Julia sets corresponding to these parameters are all Sierpinski curves, but any two such maps whose parameters are drawn from different open sets have non-conjugate dynamics. Moreover, in this particular case the Julia sets of $f_\lambda$ converge to the unit disk when $\lambda \to 0$ (see [DG06]). The cases when $d = 1$ and $n$ is arbitrary are also very different and are still under investigation.
Figure 1.1: The McMullen domain and a Cantor set of simple closed curves.
Left: the parameter \( \lambda \)-plane for the family \( f_\lambda(z) = z^4 + \lambda/z^4 \). The small disk around the origin is the McMullen domain. The colored regions are sets of parameters \( \lambda \) for which the critical orbits escape to infinity. Right: the Julia set of \( f_\lambda(z) = z^4 + 0.04/z^4 \) is a Cantor set of simple closed curves. The colored region is the basin of attraction of infinity.

See Figures 1.1 and 1.2 for examples of some of the cases described in the Escape Trichotomy theorem.

When \( |a|, |b| \neq 0,1 \) the symmetries mentioned above are lost and when \( |\lambda| < < 1 \) the effect of adding the poles no longer affect the existence of the attracting fixed point at the origin. Then, we expect the results to be very different. Indeed, we may find \( \delta_1 > 0 \) such that, if \( 0 < |\lambda| < \delta_1 \), \( f_\lambda \) still has an attracting fixed point \( q = q_\lambda \) near the origin. Let \( Q = Q_\lambda \) denote the immediate basin of attraction of \( q \). We may also find a \( \delta_2 > 0 \) such that, if \( |\lambda| < \delta_2 \), then there is a simple closed curve that separates \( B \) from \( Q \), lies near the unit circle and does not contain any of the poles of \( f_\lambda \). We denote this curve by \( \Gamma \). When the poles \( a \) and \( b \) are inside the unit circle, \( \Gamma \) is the boundary of \( B \) and we denote it by \( \partial B \). When the poles are outside the unit circle, the curve is the boundary of \( Q \) and we denote it by \( \partial Q \). Let \( \lambda_* = \min\{\delta_1, \delta_2\} \).
Figure 1.2: Sierpinski holes and Sierpinski curves.
Left: the parameter $\lambda$-plane for the family $f_\lambda(z) = z^2 + \lambda/z^2$. The red disks are Sierpinski holes. Right: the Julia set of $f_\lambda(z) = z^2 - 1/(16z^2)$ is a Sierpinski curve.

Throughout this dissertation we assume that $|\lambda| < \lambda_*$.

In spite of the large number of possibilities, when $|\lambda| << 1$ the Julia set of $f_\lambda$ is of the following kind.

**Theorem 1.0.1 (Structure of the Julia set).** Let $n \geq 2$, $d \geq 1$ and suppose that the poles $a, b$ of $f_\lambda$ satisfy $|a|, |b| \neq 0, 1$. Then, for $\lambda$ sufficiently small, the Julia set of $f_\lambda$ is composed of a countable number of disjoint simple closed curves and uncountably many point components that accumulate on these curves. Only one of these simple closed curves surrounds the origin, i.e., the curve $\Gamma$.

Although the Julia set is of one topological type, the Fatou set is of four different types depending on the position and order of the poles $a$ and $b$. We summarize the different cases in the next theorem.

**Theorem 1.0.2 (Structure of the Fatou set).** Let $n \geq 2$, $d \geq 1$ and suppose that $|a|, |b| \neq 0, 1$. Then, for $\lambda$ sufficiently small,
1. If \(|a|, |b| < 1\) and \(b \neq a^n\) (or \(b = a^n\) and \(d_a > d_b - 1\)) then \(B\) and all of its preimages are simply connected and \(Q\) is infinitely connected.

2. If \(|a| < 1\) and \(b = a^n\) with \(d_a < d_b - 1\) then \(B\) is simply connected and \(Q\) as well as its preimages are infinitely connected. Some preimages of \(B\) are disks and some are annuli.

3. If the poles \(a\) and \(b\) are outside the unit circle, then \(Q\) and all of its preimages are simply connected and \(B\) is infinitely connected.

4. If one of the poles is inside the unit circle and the other one outside, then every component of the Fatou set is infinitely connected.

A very interesting bifurcation occurs when we fix \(n, |a| < 1, d_a\) and \(d_b\) with \(d_a < d_b - 1\) and let \(b\) vary inside the unit circle. As long as \(b \neq a^n\) the preimages of \(B\) are disjoint disks bounded by simple closed curves contained inside \(\partial Q\). When \(b = a^n\) there is a preimage of the trap door \(T_b\) that is an annulus that surrounds the pole \(a\).

When \(|a| < 1, b = a^n\) and \(d_a = d_b - 1\) the critical points around \(a\) are mapped close to the boundary of \(T_b\) and we can no longer consider their orbits in a group as “one” critical orbit. This particular case can be considered as the case when \(a, b\) or both lie in the unit circle, and are not considered in this dissertation.

We can use symbolic dynamics to describe the behavior of \(f_\lambda\) on its Julia set. Let \(Q(z) = (z - a)^{d_a}(z - b)^{d_b}\) then we have,

**Theorem 1.0.3 (Dynamics on the Julia set).** Fix \(n \geq 2\) and \(d \geq 1\). Suppose \(Q_1(z)\) and \(Q_2(z)\) are in one of the four classes distinguished in Theorem 1.0.2 (if \(a = b\) for \(Q_1\) we require that also \(a = b\) for \(Q_2\)) but the exact position of the roots is arbitrary. Then there exists \(\epsilon\) such that, if \(|\lambda| < \epsilon\), the maps \(f_{\lambda,1}\) (i.e., \(f_\lambda\) perturbed with \(Q_1(z)\))
Figure 1.3: Evolution of the McMullen Domain.

Examples of Julia sets for the family $f_\lambda(z) = z^n + \lambda/(z - a)^d$. The Julia set consists of countably many simple closed curves and uncountably many points accumulating on each one of these curves. The Fatou set consists of countably many disks and one infinitely connected components. Top-left: the pole $a = -0.5$ lies inside the unit circle with $n = 3, d = 2$ and $\lambda = .005e^{ix/3}$. The white region is $\mathcal{Q}$. Top-right: magnification around the pole $a$. Bottom-left: the pole $a = 1.2i$ lies outside the unit circle with $n = 2, d = 3$ and $\lambda = .0004$. Bottom-right: magnification around the pole $a$. The white disks are $\mathcal{Q}$ and its preimages.
and $f_{\lambda^2}$ (i.e., $f_{\lambda}$ perturbed with $Q_2(z)$) are conjugate on their Julia sets. Moreover, the dynamics are determined by a specific quotient of a subshift of finite type.

For the parameter $\lambda$-plane we have the following result.

**Theorem 1.0.4 (Parameter $\lambda$-plane).** For a fixed function $f_{\lambda}(z)$, there is a punctured neighborhood $N$ of $\lambda = 0$ in parameter $\lambda$-plane in which the structure of each Julia set is as described in Theorem 1.0.1 and such that all values of $\lambda$ in this neighborhood yield maps that are conjugate to each other.

Let $a = b$ with $|a| \neq 1$. The above theorems indicate that a major change occurs in the parameter planes for these families when $a$ becomes nonzero. Let $\Lambda_a$ denote the parameter plane (the $\lambda$-plane) for a fixed value of $a$. When $a = 0$ and $1/n + 1/d < 1$, the origin in $\Lambda_0$ is surrounded by an open disk $M_0$ such that, if $\lambda \in M_0$, then the Julia set of $f_{\lambda}$ consists of uncountably many disjoint simple closed curves, each of which surrounds the origin. The set $M_0$ is the McMullen domain. But when $a \neq 0$, the origin in $\Lambda_a$ is now surrounded by a disk $N_a$ such that, if $\lambda \in N_a$, then the Julia set of $f_{\lambda}$ contains only countably many disjoint simple closed curves, only one of which surrounds the origin, and all other curves bound disjoint disks. However, the McMullen domain does continue to exist but its structure is quite different. We have,

**Theorem 1.0.5** Let $a = b \neq 0$ with $|a| \neq 1$ and let $1/n + 1/d < 1$. When $|a|$ is sufficiently small, there is an annular region that surrounds $N_a$ in which the Julia set is a Cantor set of simple closed curves each of which surrounds the origin and the pole $a$.

In Chapter 2 we prove Theorems 1.0.1, 1.0.2, 1.0.3 and 1.0.4 in the case when $a = b \neq 0$ with $|a| \neq 1$ and we also prove Theorem 1.0.5. The contents of this
Figure 1.4: Singular perturbations of $z^n$ with multiple poles.

Examples of Julia sets for the family $f_\lambda(z) = z^n + \lambda/((z - a)^{d_a} (z - b)^{d_b})$. The Julia set consists of countably many simple closed curves and uncountably many points accumulating on each one of these curves. Top: the poles lie inside the unit circle with $n = d_a = 3, d_b = 4, a = -0.25(1 + i), b = -a$ and $\lambda = 0.0001$. The Fatou set consists of countably many disks and one infinitely connected component. On the right we show a magnification around the pole $a$. Bottom: the poles lie inside the unit circle and $b = a^n$ with $n = 3, d_a = 4, d_b = 6, a = 0.5i$ and $\lambda = 5 \times 10^{-9}$. The Fatou set consists of countably many disks, annuli and infinitely connected components. On the right we show a magnification around the pole $a$. 
Figure 1.5: Singular perturbations of $z^n$ with multiple poles.

Examples of Julia sets for the family $f_\lambda(z) = z^n + \lambda/((z-a)^{d_a}(z-b)^{d_b})$. The Julia set consists of countably many simple closed curves and uncountably many points accumulating on each one of these curves. Top: the poles lie outside the unit circle with $n = 2, d_a = 8, d_b = 5, a = .9(1 + i), b = -1.2$ and $\lambda = .000005$. The Fatou set consists of countably many disks and one infinitely connected component. On the right we show a magnification around the pole $a$. Bottom: one pole inside the unit circle and the other one outside with $n = 3, d_a = 3, d_b = 6, a = -1/3(-1 + i), b = -.95(1 + i)$ and $\lambda = .0001$. Every component of the Fatou set is infinitely connected. On the right we show a magnification around the pole $a$. 
chapter are contained in [DM06]. Figure 1.3 shows examples of the maps discussed in Chapter 2.

In Chapter 3 we prove Theorems 1.0.1, 1.0.2, 1.0.3 and 1.0.4 for the more general case when \(a \neq b\). The contents of this chapter are contained in [Mar06]. To summarize our findings we can say that most perturbed maps have (topologically) the same Julia set and this Julia set is different from the one obtained with the unperturbed map. Figures 1.4 and 1.5 show examples of the maps discussed in Chapter 3.

The cases discussed above do not exhaust the possibilities for the function \(f_\lambda\). We have also studied the very different case when \(b = 0, |a| > 1\) and \(1/n + 1/d_b < 1\). In this case, the Julia set is a combination of the cases described above. When \(|\lambda| \neq 0\) the superattracting fixed point at the origin is replaced by a pole and we no longer have an attracting fixed point near the origin. In Chapter 4 we prove the following results.

**Theorem 1.0.6** Let \(n \geq 2\) with \(1/n + 1/d_b < 1\). Let \(d_a \geq 1, b = 0\) and \(|a| > 1\). Then, for \(\lambda\) sufficiently small, the Julia set of \(f_\lambda\) consists of

1. an invariant Cantor set of simple closed curves, each of which surrounds the origin;

2. a countable collection of preimages of this Cantor set of circles, none of which surrounds the origin;

3. and uncountably many point components that accumulate on both sides of every curve in these Cantor sets.

In this case every component of the Fatou set of \(f_\lambda\) is infinitely connected. The case when \(b = 0\) and \(|a| \leq 1\) is very different and it is not studied in this notes. The
Figure 1.6: Singular perturbations in the McMullen domain.

Left: the Julia set of \( f_\lambda(z) = z^n + \lambda/(z^d (z - a)^d_a) \) with \( n = 4, d_b = 3, d_a = 2, a = .75(1 + i) \) and \( \lambda = .0001 \). The Julia set consists of countably many Cantor sets of simple closed curves and uncountably many points accumulating on each curve. Every component of the Fatou set is infinitely connected. Right: magnification around the pole \( a \).

differences arise because when \( |\lambda| \) is small, the new critical points that lie near the pole \( a \) do not, in general, behave “as one” as in the case when \( |a| > 1 \).

We also prove the equivalent of Theorem 1.0.3 for this particular case.

**Theorem 1.0.7** Let \( n \geq 2 \) with \( 1/n + 1/d_b < 1 \). Let \( d_a \geq 1, b = 0 \) and \( |a_1|, |a_2| > 1 \). Then there exists \( \epsilon \) such that, if \( |\lambda|, |\mu| < \epsilon \), the maps \( f_{\lambda,1}(z) = z^n + \lambda/(z^d (z - a_1)^d_a) \) and \( f_{\lambda,2}(z) = z^n + \mu/(z^d (z - a_2)^d_a) \) are conjugate on their Julia sets. Moreover, the dynamics on the Julia sets of these maps is determined by a specific quotient of a subshift of finite type.

Figure 1.6 shows an example of the maps discussed in Chapter 4. The contents of this chapter are contained in [DM07].
Chapter 2

Evolution of the McMullen Domain

In this chapter we prove Theorems 1.0.1, 1.0.2, 1.0.3 and 1.0.4 for the case when $a = b$ with $|a| \neq 0,1$ and we also prove Theorem 1.0.5. Then, the function $f_\lambda$ from Eq. (1.1) can be written as:

$$f_\lambda(z) = z^n + \frac{\lambda}{(z - a)^d}$$

where $d = d_a + d_b \geq 1$, $n \geq 2$ and $|\lambda| << 1$.

2.1 Preliminaries

A straightforward computation shows that, when $\lambda \neq 0$, $f_\lambda$ has $n + d$ critical points that satisfy the equation

$$z^{n-1}(z - a)^{d+1} = \lambda \frac{d}{n}. \tag{2.1}$$

When $\lambda = 0$, this equation has $n + d$ zeros, the origin with multiplicity $n - 1$ and $a$ with multiplicity $d + 1$. By continuity, for small enough $|\lambda|$, these roots become simple zeros of $f_\lambda^d$ that lie in the plane approximately symmetrically distributed around the
fixed point $q$ and the pole $a$. As a consequence, when $|\lambda|$ is small, $n - 1$ of the critical points of $f_\lambda$ are grouped around $q$ near the origin while $d + 1$ of the critical points are grouped around the pole $a$.

Let $c = c_\lambda$ be a critical point of $f_\lambda$. Then we have $c^n - (c-a)^{d+1} = \lambda d/n$. Dividing by $(c - a)^d$ on both sides we find $c^{n-1}(c - a) = \lambda (c - a)^{-d}(d/n)$ and so the critical value $v$ corresponding to $c$ is given by

$$f_\lambda(c) = c^n + (n/d) \cdot c^{n-1}(c - a)$$

or, equivalently,

$$v = c^{n-1}(c (1 + n/d) - a n/d). \tag{2.2}$$

Note that, as $\lambda \to 0$, the fixed point $q \to 0$ as well. From Eq. (2.2) it follows that, if $c \to 0$, then $v \to 0$. Similarly, if $c \to a$, then $v \to a^n$. Let $S_a = \{d + 1 \text{ critical points around } a\}$ and $S_q = \{n - 1 \text{ critical points around } q\}$. We have:

**Proposition 2.1.1** When $|\lambda|$ tends to zero the critical values corresponding to the critical points in $S_q$ tend $q$ and the critical values corresponding to the critical points in $S_a$ tend to $a^n$.

To describe the structure of the Julia set, we first need to give an approximate location for this set. To do this, recall that for $|\lambda| < \lambda_*$, the immediate basin of $\infty$ is bounded by a simple closed curve that lies close to the unit circle in the plane. We can actually say more:

**Proposition 2.1.2** Suppose $n \geq 2$, $d \geq 1$ and let $|z| = r$.

1. Suppose $0 < r < \min\{1, |a|\}$. Then for sufficiently small $\lambda$, $z \in \mathcal{Q}$.
2. On the other hand, if \( r > \max\{1, |a|\} \), then for sufficiently small \( \lambda, z \in B \).

**Proof:** Fix \( n, d \) and \( a \). In the first case, we have \( |z - a| \geq |a| - |z| = |a| - r > 0 \), so that
\[
|f_\lambda(z)| \leq |z|^n + \frac{|\lambda|}{|z - a|^d} \leq r^n + \frac{|\lambda|}{(|a| - r)^d}.
\]
Let \( |\lambda| < (|a| - r)^d(r - r^n) \). Then
\[
|f_\lambda(z)| < r^n + \frac{(|a| - r)^d(r - r^n)}{(|a| - r)^d} = r^n + r - r^n = r.
\]
As a consequence, \( |f_\lambda(z)| < |z| \) and so the orbit of \( z \) converges to the fixed point \( q \) near the origin. Therefore \( z \) lies in \( Q \).

For case 2, we write \( |z - a| \geq |z| - |a| = r - |a| > 0 \). Then we have
\[
|f_\lambda(z)| \geq |z|^n - \frac{|\lambda|}{|z - a|^d} \geq r^n - \frac{|\lambda|}{(r - |a|)^d}.
\]
Let \( |\lambda| < (r - |a|)^d(r^n - r) \). Then
\[
|f_\lambda(z)| > r^n - \frac{(r - |a|)^d(r^n - r)}{(r - |a|)^d} = r^n - r^n + r = r.
\]
Hence \( |f_\lambda(z)| > |z| \) and the orbit of \( z \) converges to \( \infty \) so that \( z \in B \).

\[\square\]

**Corollary 2.1.3** If \( 0 < |a| < 1 \) and \( |\lambda| \) is sufficiently small, then all of the critical values lie in \( Q \). If \( |a| > 1 \) and \( |\lambda| \) is sufficiently small, then the critical values corresponding to the critical points in \( S_q \) lie in \( Q \) while the critical values corresponding to critical points in \( S_a \) lie in \( B \).
2.2 Structure of the Julia Set for $|a| < 1$

In this section we shall prove that for any $a$ with $0 < |a| < 1$, there exists $\epsilon_a$ such that if $|\lambda| < \epsilon_a$, then the Julia set of $f_\lambda$ consists of a countable collection of simple closed curves together with an uncountable collection of point components. Only one of these curves surrounds the origin while all others bound disks that are eventually mapped onto $B$. Moreover, any two such maps are topologically conjugate on their Julia sets.

**Proposition 2.2.1** When $|\lambda|$ is sufficiently small and $0 < |a| < 1$ the trap door $T_a$ and the immediate basin of infinity $B$ are disjoint sets. Moreover, both of these sets are bounded by simple closed curves that are also disjoint. Only the boundary of $B$ surrounds the origin.

**Proof:** When $\lambda = 0$, we may choose an annular neighborhood of the unit circle given by $N_0 = \{z | \rho_1 \leq |z| \leq \rho_2\}$ on which $f_0$ is an $n$ to $1$ expanding covering map that takes $N_0$ to a larger annulus that properly contains $N_0$. Call this image annulus $N$. The only points whose orbits under the map remain for all iterations in $N$ are points on the unit circle. We may assume further that the annulus $N$ does not contain $a$. For $\lambda$ sufficiently small, we may then find a similar annular region $N_\lambda$ that is mapped as an expanding covering map onto the same annulus $N$ by $f_\lambda$. Since $f_\lambda$ is hyperbolic on $N_\lambda$, it again follows that the set of points whose orbits remain in $N_\lambda$ under $f_\lambda$ is also a simple closed curve. In addition, all points in the exterior of this curve lie in $B$. Therefore $B$ is bounded by this simple closed curve surrounding $0$.

Now the preimage of $B$, namely $T_a$, cannot intersect $N_\lambda$, for none of the points in $N_\lambda$ that lie inside $\partial B$ are mapped into $B$. By the results of the previous section, none of the critical points can lie in the closure of $T_a$, so $\partial T_a$ is mapped as a $d$ to $1$ covering onto $\partial B$ and so the boundaries of these two sets are also disjoint simple
closed curves. Also, if $|\lambda|$ is sufficiently small, 0 lies in $Q$, so $\partial T_a$ does not surround the origin.

\[
\square
\]

As shown in the previous section, we may assume that $|\lambda|$ is small enough so that all of the critical values corresponding to the critical points in $S_a$ and $S_a$ lie in $Q$. As a consequence, none of the critical values lie in $\overline{T}_a$ or in any of the preimages of this set. Therefore all of the preimages of $\overline{T}_a$ are closed disks that are mapped univalently onto $\overline{T}_a$ by some iterate of $f_{\lambda}$. Therefore we have:

**Proposition 2.2.2** All of the preimages of $\overline{B}$ are closed disks that are disjoint from one another.

Since the boundaries of these disks all map eventually to $\partial B$, each of these curves lies in the Julia set and we have produced a countable collection of simple closed curves in $J$.

Now choose $r$ so that $r < |a| < r^{1/n} < 1$. By the results of the previous section, we may choose $|\lambda|$ sufficiently small so that:

1. All points in the closed disk of radius $r$ centered at the origin lie in $Q$;
2. The preimage of the circle of radius $r$ is a simple closed curve $\tau$ close enough to the circle of radius $r^{1/n}$ so that $a$ lies inside this curve;
3. All critical points of $f_{\lambda}$ in the set $S_a$ lie between the circle of radius $r$ and the curve $\tau$, but the images of these points lie close to $a^n$ and hence inside the circle of radius $r$.

It follows that $\tau$ also lies in the basin of $q$ and separates the boundary of $T_a$ from that of $B$. Moreover, $f_{\lambda}$ is a covering map in the entire region outside $\tau$. 
The regions $\mathcal{A}, \mathcal{B}, \mathcal{D}$, and the $I_j$'s with $j = 1, \ldots, d$ when $|a| < 1$. The annulus $\mathcal{A}$ is bounded by the curves $\tau$ and $\eta$.

Now choose $s > 1$ so that the circle of radius $s$ centered at the origin lies in $B$. Denote the preimage of this circle in $B$ by $\eta$. Note that $\eta$ is a simple closed curve that is mapped $n$ to 1 to the circle of radius $s$. Let $\mathcal{A}$ denote the annulus bounded by $\eta$ and $\tau$. Since there are no critical points in $\mathcal{A}$, $f_\lambda$ takes $\mathcal{A}$ as an $n$ to 1 covering onto the annulus given by $\{z \mid r \leq |z| \leq s\}$. As in the proof of Proposition 2.2.1, the only points whose orbits lie for all time in $\mathcal{A}$ are those in $\partial B$.

Consider the annular region bounded by the circle of radius $r$ and the curve $\tau$. Since all of the critical points in this annulus are mapped inside the circle of radius $r$ centered at the origin, there is an open disk about $a$ that is mapped as a $d$ to 1 covering (except at $a$) onto the exterior of the circle of radius $r$ (excluding the point at $\infty$). Let $\mathcal{D}$ denote this region. Then all points inside the curve $\tau$ except for those in $\mathcal{D}$ are mapped inside $|z| = r$. Note that all of the critical points belong to this set. Inside $\mathcal{D}$ there is another annulus $\mathcal{B}$ that is mapped $d$ to 1 onto $\mathcal{A}$ and a total of $d$ open disks that are each mapped 1 to 1 onto $\mathcal{D}$. Denote these disks by $I_1, \ldots, I_d$. See Figure 2.1.
Since each $I_k$ is mapped univalently over the union of all of the $I_j$, the set of points whose orbits remain for all iterations in the union of the $I_j$ forms a Cantor set on which $f_\lambda$ is conjugate to the one-sided shift map on $d$ symbols. This follows from standard arguments in complex dynamics [Mil06]. This produces an uncountable number of point components in $J$. However, there are many other point components in $J$, as any point whose orbit eventually lands in the Cantor set is also in $J$. There are, however, still other points in $J$, as we show below.

To understand the complete structure of the Julia set, we show that $J$ is homeomorphic to a quotient of a subset of the space of one sided sequences of finitely many symbols. Moreover, we show that $f_\lambda$ on $J$ is conjugate to a certain quotient of a subshift of finite type on this space. Since this is true for any $a$ with $0 < |a| < 1$ and $\lambda$ sufficiently small, this will prove our main results in the case $|a| < 1$.

To begin the construction of the sequence space, we first partition the annulus $A$ into $n$ “rectangles” that are mapped over $A$ by $f_\lambda$.

**Proposition 2.2.3** There is an arc $\xi$ lying in $A$ and having the property that $f_\lambda$ maps $\xi$ 1 to 1 onto a larger arc that properly contains $\xi$ and connects the circles of radius $r$ and $s$ centered at the origin. Moreover, $\xi$ meets $\partial B$ at exactly one point, namely one of the fixed points in $\partial B$. With the exception of this fixed point, all other points on $\xi$ lie in the Fatou set.

**Proof:** Let $p = p_\lambda$ be one of the repelling fixed points on $\partial B$. Note that $p$ varies analytically with both $\lambda$ and $a$. As is well known, there is an invariant external ray in $B$ extending from $p$ to $\infty$. Define the portion of $\xi$ in $B \cap A$ to be the piece of this external ray that lies in $A$.

To define the piece of $\xi$ lying inside $\partial B$, let $U$ be an open set that contains $p$ and meets some portion of $\tau$ and also has the property that the branch of the inverse of
that fixes $p$ is well-defined on $U$. Let $f_\lambda^{-1}$ denote this branch of the inverse of $f_\lambda$. Let $w \in \tau \cap U$ and choose any arc in $U$ that connects $w$ to $f_\lambda^{-1}(w)$. Then we let the remainder of the curve $\xi$ be the union of the pullbacks of this arc by $f_\lambda^{-k}$ for all $k \geq 0$. Note that this curve limits on $p$ as $k \to \infty$.

\[ \square \]

We now partition $\mathcal{A}$ into $n$ rectangles. Consider the $n$ preimages of $f_\lambda(\xi)$ that lie in $\mathcal{A}$. Denote these preimages by $\xi_1, \ldots, \xi_n$ where $\xi_1 = \xi$ and the remaining $\xi_j$’s are arranged counterclockwise around $\mathcal{A}$. Let $A_j$ denote the closed region in $\mathcal{A}$ that is bounded by $\xi_j$ and $\xi_{j+1}$, so that $A_n$ is bounded by $\xi_n$ and $\xi_1$. By construction, each $A_j$ is mapped 1 to 1 over $\mathcal{A}$ except on the boundary arcs $\xi_j$ and $\xi_{j+1}$, which are each mapped 1 to 1 onto $f_\lambda(\xi_1) \supset \xi_1$.

Now recall that the only points whose orbits remain for all iterations in $\mathcal{A}$ are those points on the simple closed curve $\partial B$. Let $z \in \partial B$. We may attach a symbolic sequence $S(z)$ to $z$ as follows. Consider the $n$ symbols $\alpha_1, \ldots, \alpha_n$. Define $S(z) = (s_0s_1s_2\ldots)$ where each $s_j$ is one of the symbols $\alpha_1, \ldots, \alpha_n$ and $s_j = \alpha_k$ if and only if $f_\lambda^j(z) \in A_k$. Note that there are two sequences attached to $p$, the sequences $(\overline{\alpha_1})$ and $(\overline{\alpha_n})$. Similarly, if $z \in \xi_k \cap \partial B$, then there are also two sequences attached to $z$, namely $(\alpha_{k-1}\overline{\alpha_n})$ and $(\alpha_k\overline{\alpha_1})$. Finally, if $f_\lambda^j(z) \in \xi_k$, then there are again two sequences attached to $z$, namely $(s_0s_1\ldots s_{j-1}\alpha_{k-1}\overline{\alpha_n})$ and $(s_0s_1\ldots s_{j-1}\alpha_k\overline{\alpha_1})$.

Note that if we make the above identifications in the space of all one-sided sequences of the $\alpha_j$’s, then this is precisely the same identifications that are made in coding the itineraries of the map $z \mapsto z^n$ on the unit circle. So this sequence space with these identifications and the usual quotient topology is homeomorphic to the unit circle and the shift map on this space is conjugate to $z \mapsto z^n$.

We now extend this symbolic dynamics to the set of points in the annulus $\mathcal{B}$ that are mapped to $\partial B$. Recall that $f_\lambda$ takes $\mathcal{B}$ onto $\mathcal{A}$ as a $d$ to 1 covering. So we let
Figure 2.2: A pole inside the unit circle and magnification.

Left: the Julia set of $f_\lambda(z) = z^n + \lambda/(z - a)^d$ for $n = d = 4$, $a = .5e^{i\pi/4}$ and $\lambda = 0.0001e^{i\pi/3}$. The colored regions are preimages of $B$. Right: a magnification of a region around the pole $a$.

$v_1, \ldots, v_d$ be the $d$ preimages of $\xi_1$ in $B$ arranged in counterclockwise order beginning at some given preimage. Define the rectangles $B_j, j = 1, \ldots, d$ to be the region in $B$ bounded by $v_j$ and $v_{j+1}$ where $B_d$ is bounded by $v_d$ and $v_1$. We introduce $d$ new symbols $\beta_1, \ldots, \beta_d$ and define the itinerary $S(z)$ for a point $z \in B$ that is mapped to $\partial B$ as follows. $S(z)$ is a sequence that begins with a single $\beta_k$ and then ends in any sequence of $\alpha_j$'s.

From the above, we need to make the following additional identifications in this larger set of sequences:

1. The sequences $(\beta_1 \overline{\alpha_1})$ and $(\beta_d \overline{\alpha_n})$;

2. The sequences $(\beta_{k-1} \overline{\alpha_m})$ and $(\beta_k \overline{\alpha_1})$.

Finally, we extend the definition of $S(z)$ to any point in $J$ that remains in the union of the $I_j$ by introducing the symbols 1, $\ldots, d$ and defining $S(z)$ in the usual
manner. This defines the itinerary of any point in $J$ whose orbit:

1. Remains in $\mathcal{A}$ for all iterations;

2. Starts in $\mathcal{B}$ and then remains for all subsequent iterations in $\mathcal{A}$;

3. Remains in $\mathcal{U} \cup I_j$ for all iterations.

To extend this definition to all of $J$, we let $\Sigma'$ denote the set of sequences involving the symbols $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_d, 1, \ldots, d$ subject to the following restrictions:

1. Any symbol can follow $\alpha_j$;

2. The symbol $\beta_j$ can only be followed by $\alpha_1, \ldots, \alpha_n$;

3. The symbols $1, \ldots, d$ can only be followed by $1, \ldots, d, \beta_1, \ldots, \beta_d$.

Let $\Sigma$ denote the space $\Sigma'$ where we extend the above identifications to any pair of sequences that ends in a pair of sequences identified earlier. For example, we identify the two sequences $s_0s_1 \ldots s_n\alpha_{k-1}\bar{\alpha}_n$ and $s_0s_1 \ldots s_n\alpha_k\bar{\alpha}_1$. We endow $\Sigma$ with the quotient topology. Then, by construction, the Julia set of $f_\lambda$ is homeomorphic to $\Sigma$ and $f_\lambda|J$ is conjugate to the shift map on $\Sigma$. This proves the main results in the case $|a| < 1$.

Figure 2.2 shows an example of the filled Julia set of $f_\lambda$ when $n = d = 4$ and $|a| < 1$.

### 2.3 Structure of the Julia Set for $|a| > 1$

There is a special kind of symmetry between the cases $0 < |a| < 1$ and $|a| > 1$. In Proposition 2.2.1 we proved that when $|\lambda| << 1$ the trap door $T_\alpha$ and the immediate basin of infinity $B$ were disjoint. This implied that all the preimages of $B$ were disjoint
and also implied the existence of uncountably many components accumulating on the
boundary of those preimages of $B$. Using the same argument it is easy to see that
for $|a| > 1$ and $|\lambda| << 1$ the immediate basin of attraction of $q$ and its preimages are
disjoint as well.

When $|a| > 1$ there are only $n - 1$ critical points in $Q$ so there must be $d$ preimages
of $Q$ somewhere else. By Proposition 2.1.2 these preimages cannot lie outside a circle
of radius greater than $|a|$. We shall prove that $B$ is infinitely connected and the Julia
set is made up of a countable number of simple closed curves and uncountably many
point components that accumulate on these curves.

By Corollary 2.1.3 the critical values corresponding to the critical points in $S_a$ lie
in $B$, so we next prove that the set $S_a$ itself lies in $B$.

**Proposition 2.3.1** When $|a| > 1$ and $|\lambda|$ is sufficiently small the critical points
surrounding $a$, i.e., $S_a$, are contained in $B$ and $B$ is not simply connected.

**Proof:** Fix $n \geq 2$, $d \geq 1$ and $a$ with $|a| > 1$. Fix $\rho$ with $|a| < \rho < |a|^n$ and let $\lambda$ be
sufficiently small so that:

1. $f_\lambda$ has an attracting fixed point $q$ near the origin;

2. The $n - 1$ critical points surrounding $q$ are in $Q$;

3. There is a preimage of the circle of radius $\rho$, a curve that we call $\gamma$, that lies
   completely inside the circle of radius $|a|$.

For $\lambda$ small enough $S_a$ lies close to $a$. Thus we have a function of degree $\delta = n + d$
that maps the exterior of $\gamma$ (connectivity $m$) to the exterior of the circle of radius $\rho$
(connectivity $t = 1$) and there are $N = (d + 1) + (d - 1) + (n - 1) = 2d + n - 1$ critical
points in the domain \((S_a, a \text{ and } \infty)\). By the Riemann-Hurwitz formula, we get

\[ m - 2 = \delta(t - 2) + N \quad \text{so that} \quad m = d + 1 \quad (2.3) \]

Thus, there are \(d\) holes bounded by simple closed curves in the annulus bounded by \(\gamma\) and the circle of radius \(\rho\) that contain preimages of the interior of \(\gamma\). It follows that these holes contain the \(d\) other preimages of \(Q\) so \(B\) is not simply connected.

\[ \Box \]

As before (see Propositions 2.2.1 and 2.2.2), it follows that there must be other components of the Julia set (uncountably many) accumulating on the boundaries of the preimages of \(Q\). However, the symbolic dynamics on the Julia set is somewhat different.

To begin the construction of the sequence space we first choose \(|\lambda|\) sufficiently small so that

1. All critical points in \(S_a\) lie in the annulus between the circle of radius \(\rho\) and the curve \(\gamma\);

2. All critical values that correspond to \(S_a\) lie outside the circle of radius \(\rho\).

There are \(d\) preimages of the circle of radius \(\rho\) that we call \(I_1, \ldots, I_d\) and lie inside the annulus bounded by \(\gamma\) and the circle of radius \(\rho\). Each \(I_j\) is mapped 1 to 1 to the circle of radius \(\rho\). The set of points whose orbits remain for all iterations in the union of the \(I_j\) forms a Cantor set on which \(f_\lambda\) is conjugate to the one-sided shift map on \(d\) symbols. This produces an uncountable number of point components in \(J\). However, as in the case \(|a| < 1\), there are many other point components in \(J\).

Consider a circle of radius \(s < 1\) that lies completely inside \(Q\) and contains all the critical points in \(S_u\). Call \(A\) the annulus between this circle and the curve \(\gamma\). Then,
The region $\mathcal{A}$, and the $I_j$'s for $j = 1, \ldots, d$ when $|a| > 1$. The annulus $\mathcal{A}$ is bounded by the circle of radius $s < 1$ and the curve $\gamma$ and contains the boundary of $Q$.

$\partial Q \subset \mathcal{A}$ and notice that each $I_j$ contains a copy of $\overline{Q}$ and a copy of each one of the $I_j$'s. See Figure 2.3.

As we did in the case $|a| < 1$ (see Proposition 2.2.3) we can find an arc $\xi$ lying in $\mathcal{A}$ and having the property that $f_\lambda$ maps $\xi$ to 1 onto a larger arc that properly contains $\xi$ and connects the circle of radius $s$ and $\gamma$. Moreover, $\xi$ meets $\partial Q$ at exactly one point, namely one of the fixed points in $\partial Q$. With the exception of this fixed point, all other points on $\xi$ lie in the Fatou set.

Similarly, we now partition $\mathcal{A}$ into $n$ rectangles using the $n$ preimages of $f_\lambda(\xi)$ that lie in $\mathcal{A}$. Again denote these preimages by $\xi_1, \ldots, \xi_n$ where $\xi_1 = \xi$ and the remaining $\xi_j$'s are arranged counterclockwise around $\mathcal{A}$. Let $A_j$ denote the closed region in $\mathcal{A}$ that is bounded by $\xi_j$ and $\xi_{j+1}$, so that $A_n$ is bounded by $\xi_n$ and $\xi_1$. By construction, each $A_j$ is mapped 1 to 1 over $\mathcal{A}$ except on the boundary arcs $\xi_j$ and $\xi_{j+1}$, which are each mapped 1 to 1 onto $f_\lambda(\xi_1) \supset \xi_1$. As before the only points whose orbits remain for all iterations in $\mathcal{A}$ are those points on the simple closed curve $\partial Q$. Let $z \in \partial Q$. We may attach a symbol sequence $S(z)$ to $z$ as follows. Consider the $n$ symbols $\alpha_1, \ldots, \alpha_n$. Define $S(z) = (s_0s_1s_2\ldots)$ where each $s_j$ is one of the symbols
Figure 2.4: A pole outside the unit circle and magnification.

Left: the filled Julia set of $f_\lambda(z)$ for $n = d = 4$, $a = 1.2e^{i\pi/4}$ and $\lambda = 0.0001e^{i4\pi/3}$. The colored region is $B$. The different shades represent different escape times. The white regions are $Q$ and its preimages. The picture is centered at the origin. Right: a magnification of a region around the pole $a$.

$\alpha_1, \ldots, \alpha_n$ and $s_j = \alpha_k$ if and only if $f_\lambda^{s_j}(z) \in A_k$. We use the same identifications for points in $\partial Q$ as we used for points in $\partial B$ when $|a| < 1$. Then, $f_\lambda|\partial Q$ is conjugate to $z \to z^n$ on the unit circle.

Finally, we extend the definition of $S(z)$ to any point in $J$ that remains in the union of the $I_j$’s by introducing the symbols $1, \ldots, d$ and defining $S(z)$ in the usual manner. We identify the sequences of the form $(j\alpha_1)$ and $(j\bar{\alpha}_n)$ as well as $(j\alpha_k\bar{\alpha}_n)$ and $(j\alpha_k\bar{\alpha}_1)$.

Let $\Sigma'$ denote the set of sequences $\alpha_1, \ldots, \alpha_n, 1, \ldots, d$. Let $\Sigma$ denote the space $\Sigma'$ with all of the identifications described above and endow $\Sigma$ with the quotient topology. Then, by construction, the Julia set of $f_\lambda$ is homeomorphic to $\Sigma$ and $f_\lambda|J$ is conjugate to the full shift map on $\Sigma$. This proves the main results in the case $|a| > 1$. 
Figure 2.4 shows an example of the filled Julia set of $f_\lambda(z)$ with $n = d = 4$ when the pole $a$ is outside the unit circle.

2.4 The $a\lambda$-plane

The theorems we have proved describe the structure of the $a\lambda$-plane when $|a| \neq 1$ and $|\lambda|$ is small. When $a = 0$, there is a McMullen domain surrounding $\lambda = 0$. But when $a \neq 0$, the McMullen domain is replaced by a region around $\lambda = 0$ where the Julia sets are quite different. Actually, the McMullen domain no longer surrounds the origin in the $\lambda$-plane, but rather has evolved into a different type of set that contains an annulus surrounding 0. Figure 2.5 shows schematically two different regions in an $a\lambda$-slice of the parameter space when $1/n + 1/d < 1$. The darker region around the $\lambda$ axis (or plane) when $a = 0$ contains parameters in the McMullen domains for which the Julia sets of $f_\lambda$ are Cantor sets of simple closed curves. As we separate from the $\lambda$ plane, that is, as $a$ becomes nonzero but $|\lambda|$ is very small, we find the region for which the Julia sets of $f_\lambda$ are the sets described in Theorem 1.0.1.

Note that if $a = 0$ and $\lambda$ lies in the McMullen domain, then the map $f_\lambda$ is hyperbolic on its Julia set. So the Cantor set of circles persists as we vary both $a$ and $\lambda$ starting at this parameter. In particular, there exists $\delta_\lambda$ such that, if $|a| < \delta_\lambda$, then the Julia set of $f_\lambda$ is a Cantor set of circles. Now for any given $\tilde{\lambda}$ in the McMullen domain, we may choose $\delta_\lambda$ so that this phenomenon persists for all $a$ and $\lambda$ in a neighborhood of $a = 0$ and $\lambda = \tilde{\lambda}$. Thus for any simple closed curve $\gamma$ in the McMullen domain for $a = 0$, there exists $a_\gamma$ such that if $|a| < a_\gamma$ and $\lambda$ lies on $\gamma$, then again $J$ is a Cantor set of simple closed curves. Therefore, if we fix $a$ with $|a| < a_\gamma$, then there is an annulus in the corresponding $\lambda$-plane with Julia sets of this type. Thus we see that, as $a$ moves away from 0, the McMullen domain does not disappear.
Figure 2.5: The $a\lambda$-plane.

A schematic picture of a slice of the $a\lambda$-space where $|\lambda| < < 1$ and $1/n + 1/d < 1$. The darker regions along the $\lambda$ axis represent McMullen domains in $\lambda$-plane. The shaded regions around the $a$ axis show where we find the Julia sets studied in this dissertation.

Rather, a hole opens up around $\lambda = 0$ in which the Julia set is as described in Section § 2.2 while the McMullen domain moves away from the origin. Between these two regions, the Julia sets clearly undergo major transformations, so the structure of the parameter planes in these intermediate regions is an open problem.
Chapter 3

Singular Perturbations of \( z^n \) with Multiple Poles

In this chapter we prove Theorems 1.0.1, 1.0.2, 1.0.3 and 1.0.4 for the case when we have two poles. That is, we study the function

\[
f_\lambda(z) = z^n + \frac{\lambda}{(z - a)^{d_a} (z - b)^{d_b}}
\]

where \( n, d_a \) and \( d_b \) are integers such that \( n \geq 2 \) and \( d_a, d_b \geq 1 \), the poles \( a \) and \( b \) are such that \( |a|, |b| \neq 0,1 \) and \( \lambda \) is a sufficiently small complex parameter. Despite the large number of possibilities there are only four different cases that correspond to different positions and orders of the poles \( a \) and \( b \).

3.1 Preliminaries

Let \( f_\lambda(z) = z^n + \lambda/Q(z) \) where \( Q(z) = (z - a)^{d_a} (z - b)^{d_b} \), \( n \geq 2 \) and \( d_a, d_b \geq 1 \). The degree of \( Q(z) \) is \( d = d_a + d_b \). A straightforward calculation shows that the critical
points $c = c_\lambda$ of $f_\lambda$ satisfy

$$nc^{n-1}(c - a)^{2d_a} (c - b)^{2d_b} = \lambda \left(d_a (c - a)^{d_a-1} (c - b)^{d_b} + d_b (c - b)^{d_b-1} (c - a)^{d_a}\right).$$

This equation is trivially satisfied by $c = a, b$ with orders $d_a - 1$ and $d_b - 1$, respectively. If we remove these roots we can write

$$nc^{n-1}(c - a)^{d_a + 1} (c - b)^{d_b + 1} = \lambda \left(d_a (c - b) + d_b (c - a)\right). \quad (3.1)$$

When $\lambda = 0$ this equation has $n + d + 1$ roots, the origin with multiplicity $n - 1$, the pole $a$ with multiplicity $d_a + 1$ and the pole $b$ with multiplicity $d_b + 1$. By continuity, for small enough $|\lambda|$, these roots become simple zeros of $f_\lambda^\prime$ that lie in the plane approximately symmetrically distributed around the fixed point $q$ near the origin and the poles $a$ and $b$. When $\lambda \to 0$ then either $c \to 0$, $c \to a$ or $c \to b$.

As a consequence, when $|\lambda|$ is small, $n - 1$ of the critical points of $f_\lambda$ are grouped around $q$ while $d_a + 1$ critical points are grouped around the pole $a$ and $d_b + 1$ are grouped around $b$.

The critical value corresponding to $c$ is given by $v = f_\lambda(c) = c^n + \lambda/Q(c)$ and using Eq. (3.1) we get

$$v = c^n + \frac{nc^{n-1}(c - a)(c - b)}{d_a (c - b) + d_b (c - a)}. \quad (3.2)$$

Notice that when $\lambda \to 0$, the fixed point $q \to 0$, too. From Eq. (3.2) it follows that, if $c \to 0$, then $v \to 0$. Similarly, if $c \to a$, then $v \to a^n$ and if $c \to b$ then $v \to b^n$.

Let $S_a = \{d_a + 1$ critical points around $a\}$, $S_b = \{d_b + 1$ critical points around $b\}$ and $S_q = \{n - 1$ critical points around $q\}$. We have:

**Proposition 3.1.1** When $|\lambda|$ tends to zero the critical values corresponding to the
critical points in $S_q$ tend to $q$ (near the origin), the critical values corresponding to
the critical points in $S_a$ tend to $a^n$ and, the critical values corresponding to the critical
points in $S_b$ tend to $b^n$.

To describe the structure of the Julia set, we first need to give an approximate
location for this set. To do this, recall that for $|\lambda| < \lambda_s$, the immediate basin of $\infty$
and the immediate basin of $Q$ are separated by a simple closed curve that we call $\Gamma$
and lies close to the unit circle in the plane. We can actually say more:

**Proposition 3.1.2** Suppose $n \geq 2$, $d_a, d_b \geq 1$, $|a|, |b| \neq 0, 1$, let $|z| = r$ and let
\[ a_{\min} = \min\{|a|, |b|\} \text{ and } a_{\max} = \max\{|a|, |b|\}. \]

1. Suppose $0 < r < \min\{1, a_{\min}\}$. Then for sufficiently small $\lambda$, $z \in Q$.

2. On the other hand, if $r > \max\{1, a_{\max}\}$, then for sufficiently small $\lambda$, $z \in B$.

**Proof:** Fix $n$ and $a, b$ together with $d_a, d_b$. In the first case, we have $|z - a| \geq
|a| - |z| \geq a_{\min} - r > 0$. Similarly, $|z - b| \geq |b| - |z| \geq a_{\min} - r > 0$ so that

\[ |f_\lambda(z)| \leq |z|^n + \frac{|\lambda|}{|z - a|^{d_a}|z - b|^{d_b}} \leq r^n + \frac{|\lambda|}{(a_{\min} - r)^d}. \]

Let $|\lambda| < (a_{\min} - r)^d(r - r^n)$. Then

\[ |f_\lambda(z)| < r^n + \frac{(a_{\min} - r)^d(r - r^n)}{(a_{\min} - r)^d} = r^n + r - r^n = r. \]

As a consequence, $|f_\lambda(z)| < |z|$ and so the orbit of $z$ converges to the fixed point $q$
near the origin. Therefore $z$ lies in $Q$.

For case 2, we write $|z - a| \geq |z| - |a| \geq r - a_{\max} > 0$. Similarly, $|z - b| \geq$
$|z| - |b| \geq r - a_{\text{max}} > 0$. Then we have

$$|f_\lambda(z)| \geq |z|^n - \frac{|\lambda|}{|z - a||z - b|^d} \geq r^n - \frac{|\lambda|}{(r - a_{\text{max}})^d}.$$

Let $|\lambda| < (r - a_{\text{max}})^d(r^n - r)$. Then

$$|f_\lambda(z)| > r^n - \frac{(r - a_{\text{max}})^d(r^n - r)}{(r - a_{\text{max}})^d} = r^n - r^n + r = r.$$

Hence $|f_\lambda(z)| > |z|$ and the orbit of $z$ converges to $\infty$ so that $z \in B$.

$\square$

To study the cases when $b = a^n$ we need also consider the rate at which the preimages of $\Gamma$ near $b$ approach $b$, the rate at which the critical points in $S_a$ approach $a$ and the rate at which the critical values corresponding to the critical points in $S_a$ approach $b$ as $\lambda \to 0$.

To simplify the notation we define

$$\phi(a) = \left(\frac{\lambda d_a}{na^{n-1}(a - b)^{d_b}}\right)^{1/(d_a + 1)}.$$

**Proposition 3.1.3** When $|\lambda|$ is sufficiently small we have that:

1. the critical points $c$ in $S_a$ are such that $c \approx a + \phi(a)$ and,

2. the critical values $v$ corresponding to $S_a$ are such that $v \approx a^n + k\lambda^{1/(d_a + 1)}$ where $k$ is a constant independent of $\lambda$.

A similar statement follows for the critical points in $S_b$ and their critical values.

**Proof:** For the first part let $z = a + \phi(a)$ and replace $c$ by $z$ in Eq. (3.1) to get

$$\frac{\lambda d_a}{na^{n-1}(a - b)^{d_b}}(a - b + \phi(a))^{(d_b+1)}n(a + \phi(a))^{n-1} = \lambda(d_a(a - b + \phi(a)) + d_b\phi(a)),$$
Cancel $\lambda$ on both sides and $n$ on the left hand side, then let $\lambda \to 0$. It is easy to see that when $\lambda \to 0$ then $\phi(a) \to 0$. So, when $\lambda$ is sufficiently small we can simplify to get the identity

$$d_a(a - b) = d_a(a - b).$$

Hence, for $|\lambda|$ small enough the critical points in $S_a$ can be written as $c \approx a + \phi(a)$, as we wanted to show.

For the second part let $z$ be as in part 1. Then apply $f_\lambda$ to get

$$v = f_\lambda(c) \approx f(z) = (a + \phi(a))^n + \frac{\lambda}{\phi(a)^{d_b} (a - b + \phi(a))^{d_b}}.$$ 

Let $\lambda \to 0$. Since $d_a, d_b \geq 1$ we get the result.

Notice that Proposition 3.1.3 implies that when $\lambda \to 0$ the critical points in $S_a$ and the critical values corresponding to those critical points converge at the same rate proportional to $\lambda^{1/(d_a+1)}$ to $a$ and to $a^n$, respectively. The same is true for the critical points in $S_b$ and their critical values that converge at a rate proportional to $\lambda^{1/(d_b+1)}$ to $b$ and $b^n$, respectively.

**Proposition 3.1.4 (Size of $T_b$)** When $\lambda$ is sufficiently small and the pole $b$ is inside the unit circle, the trap door $T_b$ is bounded by a simple closed curve that is close to the curve $\gamma_b(t)$ given by

$$(z(t) - b)^{d_b} = \frac{\lambda}{(e^{\mu} - b^n)(b - a)^{d_a}}$$

parametrized by $t$ where $0 \leq t \leq 2\pi$ and oriented in the clockwise direction. A similar statement follows for $T_a$. 
Proof: The result follows from the following facts:

1. \( \gamma_b(t) \) is a simple closed curve with winding number equal to 1 around \( b \) and,
2. when \( \lambda \to 0 \) the image of \( \gamma_b(t) \to \{ z : |z| = 1 \} \).

Consider the case \( d_b = 1 \). The curve \( \gamma_b(t) \) depends continuously on \( t \) and \( \gamma_b(0) = \gamma_b(2\pi) \) but \( \gamma_b(t_1) \neq \gamma_b(t_2) \) for all \( t_1, t_2 \neq 0, 2\pi \). Let \( \eta(\gamma_b(t), b) \) be the winding number of \( \gamma_b(t) \) with respect to \( b \) and let \( \eta(e^{it}, b^n) \) be the winding number of the unit circle with respect to \( b^n \). Since \( |b| < 1 \) it follows that \( \eta(e^{it}, b^n) = 1 \). A simple computation shows that

\[
\eta(\gamma_b(t), b) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ie^{it}}{e^{it} - b^n} dt = \eta(e^{it}, b^n).
\]

Then \( \eta(\gamma_b(t), b) = 1 \) as we wanted to show. When \( d_b > 1 \) consider the curve

\[
\gamma(t) = \frac{\lambda}{(e^{it} - b^n)(b - a)^{d_b}}.
\]

Again, notice that \( \eta(\gamma(t), 0) = \eta(e^{it}, b^n) = 1 \) and so \( \gamma(t) \) is a simple closed curve that surrounds the origin. Then, when we consider \( (z(t) - b)^{d_b} = \gamma(t) \) we have that the union of the different \( d_b \) subarcs obtained from the \( d_b \) branches of \( z(t) = b + \gamma(t)^{1/d_b} \) make up the curve \( \gamma_b(t) \) that surrounds \( b \) and part 1 follows.

For part 2 let

\[
z(t) = b + \left( \frac{\lambda}{(e^{it} - b^n)(b - a)^{d_n}} \right)^{1/d_b}
\]

Then, compute \( f_\lambda(z(t)) \) to get

\[
\left( b + \left( \frac{\lambda}{(e^{it} - b^n)(b - a)^{d_n}} \right)^{1/d_b} \right)^n + \frac{\lambda}{(e^{it} - b^n)(b - a)^{d_n}} \left( b + \left( \frac{\lambda}{(e^{it} - b^n)(b - a)^{d_n}} \right)^{1/d_b} - a \right)^{d_n}.
\]

Cancel \( \lambda \) in the second term and let \( \lambda \to 0 \). We get \( f_\lambda(z(t)) = b^n + (e^{it} - b^n) = e^{it} \) as
we wanted to show.

Proposition 3.1.4 implies that when \( \lambda \to 0 \) the trap door \( T_b \) shrinks at a rate proportional to \( \lambda^{1/d_b} \). Proposition 3.1.3 implies that when \( \lambda \to 0 \) the critical values corresponding to the critical points in \( S_a \) approach \( a^n \) at a rate proportional to \( \lambda^{1/(d_a+1)} \). Then, when \( |a| < 1, b = a^n \) and \( d_a < d_b - 1 \) for \( \lambda \) sufficiently small we have that the critical points in \( S_a \) are mapped inside \( T_b \). Then, the second iterate of the critical points in \( S_a \) lies in \( B \) and so their orbits converge to infinity. This fact causes major changes in the structure of the Fatou set of \( f_\lambda \).

**Remark 3.1.5** It follows as in the proof of Proposition 3.1.4 that when \( \lambda \) is sufficiently small and \( |b| > 1 \) the set defined by \( \gamma_b(t) \) consists of \( d_b \) simple closed curves that lie close to \( d_b \) preimages of \( \Gamma \). These curves are approximately symmetrically distributed around the pole \( b \) but do not contain \( b \) and when \( \lambda \to 0 \) they approach \( b \) at a rate proportional to \( \lambda^{1/d_b} \). A similar statement follows for the \( d_a \) preimages of \( \Gamma \) that lie near the pole \( a \) when \( |a| > 1 \). Then, for sufficiently small \( \lambda \), Proposition 3.1.3 implies that when \( |a| > 1, b = a^n \) and \( d_a \neq d_b - 1 \) the critical values corresponding to the critical points in \( S_a \) and \( S_b \) lie in \( B \).

We can summarize the results in the following way.

**Corollary 3.1.6** *(Location of the critical values)* If \( |\lambda| \) is sufficiently small then the critical values corresponding to the critical points in \( S_a \) lie in \( Q \) and,

1. if \( |a|, |b| < 1 \) and \( b \neq a^n \) (or \( b = a^n \) and \( d_a > d_b - 1 \)) the critical values corresponding to critical points in \( S_a \) and \( S_b \) lie in \( Q \).

2. if \( |a| < 1, b = a^n \) and \( d_a < d_b - 1 \) then the critical values corresponding to the critical points in \( S_b \) lie in \( Q \) and the critical values corresponding to the critical points in \( S_a \) lie in \( T_b \).
3. if \(|a|, |b| > 1\) then the critical values corresponding to critical points in \(S_a\) and \(S_b\) lie in \(B\).

4. if \(|a| < 1\) and \(|b| > 1\) then the critical values corresponding to the critical points in \(S_a\) lie in \(Q\) and the critical values corresponding to the critical points in \(S_b\) lie in \(B\).

It follows that when \(|\lambda| << 1\) the critical points divide in two sets, one set of critical points that converge to \(q\) near the origin and another set of critical points that converge to \(\infty\). The topological characteristics of the Julia sets of \(f_{\lambda}\) as well as the dynamics of \(f_{\lambda}\) on \(J\) are completely determined by the position and order of the poles \(a\) and \(b\).

### 3.2 Some special cases

Our goal here is to describe the setup for the general case with some illustrative examples.

#### 3.2.1 Example 1: \(|a|, |b| < 1\) and \(b \neq a^n\)

We consider the function \(f_{\lambda}(z) = z^n + \lambda/Q(z)\) where the polynomial \(Q(z)\) is given by \(Q(z) = (z - a)^4(z - b)^5\), \(n = 2\) and \(a\) and \(b\) are such that \(|a|, |b| < 1\) and \(b \neq a^n\). This example includes what happens in the case \(|a| < 1\) with \(b = a^n\) when \(d_a > d_b - 1\). Then we have

\[
f_{\lambda}(z) = z^2 + \frac{\lambda}{(z - a)^4(z - b)^5}.
\]

Notice that the degree of \(f_{\lambda}\) is 11 and so there are 20 critical points counted with multiplicity. Infinity is a critical point of order 1. Thus, there are 19 critical points
of $f_{\lambda}$ that satisfy the equation

$$2c(c - a)^5(c - b)^{10} = \lambda(4(c - a)^3(c - b)^5 + 5(c - b)^4(c - a)^4).$$

As can be seen in this equation, the pole $a$ is a critical point of order 3 and the pole $b$ is a critical point of order 4. If we remove these solutions we find

$$2c(c - a)^5(c - b)^6 = \lambda(4(c - b) + 5(c - a))$$

and we see that when $\lambda \to 0$ there is 1 critical point that approaches 0, and 5 and 6 critical points that approach $a$ and $b$, respectively. The critical values corresponding to these 12 “free” critical points are determined by

$$v = f_{\lambda}(c) = c^2 + \frac{2c(c - a)(c - b)}{4(c - b) + 5(c - a)}.$$

Note that if $c \to 0$ then $v \to 0$, if $c \to a$ then $v \to a^2$ and if $c \to b$ then $v \to b^2$.

Assume that $|\lambda|$ is very small. Then we get 12 critical points that are attracted to $q$ near the origin. In this case, there are two trap doors, $T_a$ and $T_b$, bounded by simple closed curves. These are disjoint preimages of the immediate basin of attraction of infinity $B$. The set $T_a$ is mapped 4 to 1 onto $B$ and $T_b$ is mapped 5 to 1 onto $B$. The Julia set contains the boundaries of the preimages of $B$. So, the Julia set consists of countably many simple closed curves and an uncountable number of point components that accumulate on them. Only one component of the Julia set surrounds the origin, all the other curves are disjoint and lie inside $\Gamma = \partial B$. No other curve of the Julia set contains other components of the Julia set. See Figure 3.1.
Figure 3.1: Singular perturbations with multiple poles. Example 1.

Top-left: the Julia set of \( f_\lambda(z) = z^2 + \lambda / ((z - a)^4(z - b)^5) \) when the pole \( a = 1/2, b = -1/3i \) and \( \lambda = .000001 \). The white region is \( Q \) and the colored regions are \( B \) and its preimages. Top-right: magnification about the pole \( a \). Bottom: magnification about the pole \( b \).
3.2.2 Example 2: $|a|, |b| < 1$, $b = a^n$ and $d_a < d_b - 1$

We consider the function $f_{\lambda}(z) = z^n + \lambda/Q(z)$ where the polynomial $Q(z)$ is given by $Q(z) = (z - a)^3(z - b)^7, n = 5$ and $a$ and $b$ are such that $|a| < 1$ and $b = a^n$. Then we have

$$f_{\lambda}(z) = z^5 + \frac{\lambda}{(z - a)^3(z - b)^7}.$$ 

Notice that the degree of $f_{\lambda}$ is 15 and so there are 28 critical points counted with multiplicity. Infinity is a critical point of order 4. Thus, there are 24 critical points of $f_{\lambda}$ that satisfy the equation

$$5c^4(c - a)^6(c - b)^{14} = \lambda(3(c - a)^2(c - b)^7 + 7(c - b)^6(c - a)^3).$$

As can be seen in this equation, the pole $a$ is a critical point of order 2 and the pole $b$ is a critical point of order 6. If we remove these solutions we find

$$5c^4(c - a)^4(c - b)^8 = \lambda(3(c - b) + 7(c - a))$$

and we see that when $\lambda \to 0$ there are 4 critical points that approach 0, and 4 and 8 critical points that approach $a$ and $b$, respectively. The critical values corresponding to these 16 “free” critical points are determined by

$$v = f_{\lambda}(c) = c^5 + \frac{5c^4(c - a)(c - b)}{3(c - b) + 7(c - a)}.$$ 

Note that if $c \to 0$ then $v \to 0$, if $c \to a$ then $v \to a^5$ and if $c \to b$ then $v \to b^5$.

Assume that $|\lambda|$ is very small. Then the 8 critical points around $b$ and the 4 critical points around $q$ are attracted to $q$. The 4 critical points around the pole $a$ are mapped inside the trap door $T_b$ and after one more iteration inside $B$. The preimages
Figure 3.2: Singular perturbations with multiple poles. Example 2.

Top-left: The Julia set of \( f_\lambda(z) = z^5 + \lambda / ((z - a)^3(z - b)^7) \) when the pole \( a = 3/4, b = a^3 \) and \( \lambda = .00000001 \). Top-right: magnification about the pole \( a \). Bottom-left: magnification about the pole \( b \). Bottom-right: further magnification on the boundary of the trap door around the pole \( a \).
of $T_a$ are disks bounded by simple closed curves. The preimage of $T_b$ that contains the 4 critical points around $a$ is an annulus that surrounds $a$. All the preimages of this annulus are annuli. The other 11 preimages of $T_b$ are disks. The immediate basin of attraction of $q$ is infinitely connected as well as all its preimages. The Julia set of $f_\lambda$ is composed of the boundaries of the trap doors as well as the boundaries of their preimages and uncountably many point components that accumulate on those curves. Some components of the Julia set contain infinitely many other components of $J$ and some none. The only curve in $J$ that surrounds the origin is $\Gamma$. See Figure 3.2.

### 3.2.3 Example 3: $|a|, |b| > 1$.

We consider the function $f_\lambda(z) = z^n + \lambda/Q(z)$ where the polynomial $Q(z)$ is given by $Q(z) = (z-a)^3(z-b)^4$, $n = 3$ and $a$ and $b$ are such that $|a|, |b| > 1$. Then we have

$$f_\lambda(z) = z^3 + \frac{\lambda}{(z-a)^3(z-b)^4}.$$ 

Notice that the degree of $f_\lambda$ is 10 and so there are 18 critical points counted with multiplicity. Infinity is a critical point of order 2. Thus, there are 16 critical points of $f_\lambda$ that satisfy the equation

$$3c^2(c-a)^6(c-b)^8 = \lambda (3(c-a)^2(c-b)^4 + 4(c-b)^3(c-a)^3).$$

As can be seen in this equation, the pole $a$ is a critical point of order 2 and the pole $b$ is a critical point of order 3. If we remove these solutions we find

$$3c^2(c-a)^4(c-b)^5 = \lambda (3(c-b) + 4(c-a)).$$
Figure 3.3: Singular perturbations with multiple poles. Example 3.
Top-left: the Julia set of $f_{\lambda}(z) = z^3 + \lambda/((z - a)^3(z - b)^4)$ when the pole $a = 1.2$, $b = -1.2i$ and $\lambda = 0.001$. The large white region is $Q$. The colored region is $B$. Notice the preimages of $Q$ outside the curve $\Gamma = \partial Q$. Top-right: magnification about the pole $a$. Bottom: magnification about the pole $b$. 
and we see that when $\lambda \to 0$ there are 2 critical points that approach 0, and 4 and 5 critical points that approach $a$ and $b$, respectively. The critical values corresponding to these 11 “free” critical points are determined by

$$v = f_\lambda(c) = c^3 + \frac{3c^2(c - a)(c - b)}{3(c - b) + 4(c - a)}.$$ 

Note that if $c \to 0$ then $v \to 0$, if $c \to a$ then $v \to a^3$ and if $c \to b$ then $v \to b^3$.

Assume that $|\lambda|$ is very small. Then there are 2 critical points attracted to $q$ and 9 critical points around $a$ and $b$ attracted to $\infty$. Since the degree of $f_\lambda$ is 10, there must be 7 preimages of points in $Q$ outside $\Gamma = \partial Q$. The preimages of $Q$ are bounded by disjoint simple closed curves. The Julia set of $f_\lambda$ is composed of these curves and an uncountable number of point components that accumulate on them. There is only one component curve of the Julia set that surrounds the origin, namely $\partial Q$, and the rest of the curves lie outside $\partial Q$. No curve of the Julia set contains other components of the Julia set. See Figure 3.3.

### 3.2.4 Example 4: $|a| < 1$ and $|b| > 1$.

We consider the function $f_\lambda(z) = z^n + \lambda/Q(z)$ where the polynomial $Q(z)$ is given by $Q(z) = (z - a)^5(z - b)^4$, $n = 2$ and $a$ and $b$ are such that $|a| < 1$ and $|b| > 1$. Then we have

$$f_\lambda(z) = z^2 + \frac{\lambda}{(z - a)^5(z - b)^4}.$$ 

Notice that the degree of $f_\lambda$ is 11 and so there are 20 critical points counted with multiplicity. Infinity is a critical point of order 1. Thus, there are 19 critical points of $f_\lambda$ that satisfy the equation

$$2c(c - a)^{10}(c - b)^8 = \lambda(5(c - a)^4(c - b)^4 + 4(c - b)^3(c - a)^5).$$
As can be seen in this equation, the pole $a$ is a critical point of order 4 and the pole $b$ is a critical points of order 3. If we remove these solutions we find

$$2c(c - a)^6(c - b)^5 = \lambda(5(c - b) + 4(c - a))$$

and we see that when $\lambda \to 0$ there is 1 critical point that approaches 0, 6 critical points that approach $a$ and 5 that approach $b$. The critical values corresponding to these 12 “free” critical points are determined by

$$v = f_\lambda(c) = c^2 + \frac{2c(c - a)(c - b)}{5(c - b) + 4(c - a)}.$$

Note that if $c \to 0$ then $v \to 0$, if $c \to a$ then $v \to a^2$ and if $c \to b$ then $v \to b^2$.

Assume that $|\lambda|$ is very small. Then there are 7 critical points that are attracted to $q$ and 5 critical points near $b$ that are attracted to $\infty$. The curve $\Gamma$ separates $\mathcal{Q}$ from $B$. The trap door $T_a$ is mapped 5 to 1 outside of the curve $\Gamma$. There are 4 preimages of the interior of $\Gamma$ outside $\Gamma$. These components of the Fatou set of $f_\lambda$ are all disjoint and infinitely connected. The preimages of $\Gamma$ are disjoint simple closed curves contained in the Julia set of $f_\lambda$. The Julia set contains also uncountably many point components that accumulate on these curves. Only $\Gamma$ surrounds the origin. Some of the curves lie inside $\Gamma$ and some lie outside $\Gamma$ but every curve component of the Julia set contains infinitely many curves inside itself. See Figure 3.4.

The main differences between these examples and the general case are the position and order of the poles $a$ and $b$. 
Figure 3.4: Singular perturbations with multiple poles. Example 4.
Top-left: the Julia set of $f_\lambda(z) = z^2 + \lambda/((z - a)^5(z - b)^4)$ when the pole $a = 1/2 e^{i\pi/4}$, $b = 1.2i$ and $\lambda = .00001$. Top-right: magnification about the pole $a$. Notice the preimages of $Q$ inside $\partial T_a$. Bottom-left: magnification about the pole $b$. Notice the preimages of the interior of $\Gamma$ around $b$. Bottom-right: further magnification on the boundary of the trap door around the pole $a$. 
3.3 Structure of the Julia Set for $|a|, |b| < 1$

In this section we shall prove that when $0 < |a|, |b| < 1$, there exists $\epsilon$ such that if $|\lambda| < \epsilon$, then the Julia set of $f_\lambda$ consists of a countable collection of disjoint simple closed curves together with an uncountable collection of point components. Only one of these curves surrounds the origin while all others bound regions that are eventually mapped onto $B$. Moreover, any two such maps are topologically conjugate on their Julia sets.

**Proposition 3.3.1** When $|\lambda|$ is sufficiently small and $|a|, |b| < 1$ the trap doors $T_a$ and $T_b$ and the immediate basin of infinity $B$ are disjoint sets. Moreover, all these sets are bounded by simple closed curves that are also disjoint. Only the boundary of $B$ surrounds the origin.

**Proof:** When $\lambda = 0$, we may choose an annular neighborhood $N_0$ of the unit circle given by $N_0 = \{z \mid \rho_1 \leq |z| \leq \rho_2\}$ on which $f_\lambda$ is an $n$ to 1 expanding covering map that takes $N_0$ to a larger annulus that properly contains $N_0$. Call this image annulus $N$. The only points whose orbits under the map remain for all iterations in $N$ are points on the unit circle. We may assume further that the annulus $N$ does not contain $a$ and $b$. For $\lambda$ sufficiently small, we may then find a similar annular region $N_\lambda$ that is mapped as an expanding covering map onto the same annulus $N$ by $f_\lambda$. Since $f_\lambda$ is hyperbolic on $N_\lambda$, it again follows that the set of points whose orbits remain in $N_\lambda$ under $f_\lambda$ is a simple closed curve. In addition, all points in the exterior of this curve lie in $B$. Therefore $B$ is bounded by this simple closed curve surrounding $0$.

Now the preimages of $B$, namely $T_a$ and $T_b$, cannot intersect $N_\lambda$, for none of the points in $N_\lambda$ that lie inside $\partial B$ are mapped into $B$. By the results of Section § 3.1, none of the critical points can lie in the closure of $T_a$ or in the closure of $T_b$, so $\partial T_a$ (resp. $\partial T_b$) is mapped as a $d_a$ to 1 (resp. $d_b$ to 1) covering onto $\partial B$ and
so the boundaries of these sets are also disjoint simple closed curves. Also, if $|\lambda|$ is sufficiently small, 0 lies in $Q$, so $\partial T_a$ and $\partial T_b$ do not surround the origin.

Consider first the case when $b \neq a^n$ (or $b = a^n$ but $d_a > d_b - 1$). As shown in Section § 3.1, we may assume that $|\lambda|$ is small enough so that all of the critical values corresponding to the free critical points lie in $Q$. As a consequence, none of the critical values lie in $T_a \cup T_b$ or in any of the preimages of this set. Therefore we have that all of the preimages of $T_a$ (resp. $T_b$) are closed disks that are mapped univalently onto $\overline{T}_a$ (resp. $\overline{T}_b$) by some iterate of $f_\lambda$. Therefore we have:

**Proposition 3.3.2** All of the preimages of $\overline{B}$ are closed disks that are disjoint from one another.

Since the boundaries of these disks all map eventually to $\partial B$, each of these curves lies in the Julia set and we have produced a countable collection of simple closed curves in $J$.

Now let $\rho$ be a simple closed curve that surrounds the origin and such that it contains $a^n$ and $b^n$. By the results of Section § 3.1, we may choose $|\lambda|$ sufficiently small so that:

1. All points in the closed disk bounded by $\rho$ lie in $Q$;

2. There is a preimage of the curve $\rho$, a simple closed curve $\tau$, that contains $a$ and $b$;

3. All critical points of $f_\lambda$ in the sets $S_a$ and $S_b$ lie between the curve $\rho$ and the curve $\tau$, but the images of these points lie close to $a^n$ and $b^n$ and hence inside the curve $\rho$. 
It follows that $\tau$ also lies in the basin of $q$ and separates the boundary of $T_a$ and $T_b$ from that of $B$. Moreover, $f_\lambda$ is a covering map in the entire region outside $\tau$.

Now choose $s > 1$ so that the circle of radius $s$ centered at the origin lies in $B$. Denote the preimage of this circle in $B$ by $\eta$. Note that $\eta$ is a simple closed curve that is mapped $n$ to $1$ to the circle of radius $s$. Let $A$ denote the annulus bounded by $\eta$ and $\tau$. Since there are no critical points in $A$, $f_\lambda$ takes $A$ as an $n$ to $1$ covering onto the annulus defined by $\rho$ and the circle of radius $s$. As in the proof of Proposition 3.3.1, the only points whose orbits lie for all time in $A$ are those in $\partial B$.

Consider the region bounded by the curve $\rho$ and the curve $\tau$. Since all of the critical points in this region are mapped inside $\rho$, it follows that there are open disks about the poles $a$ and $b$ that are mapped as a $d_a$ to $1$ (resp. $d_b$ to $1$) covering (except at $a$ and $b$) onto the exterior of the curve $\rho$ (excluding the point at $\infty$). Let $D_a$ and $D_b$ denote these regions. Then all points inside the curve $\tau$ except for those in $D_a$ and $D_b$ are mapped inside $\rho$. Note that all of the critical points in $S_q$, $S_a$ and $S_b$ belong to this set. Inside $D_a$ there is an annulus $B_a$ that is mapped $d_a$ to $1$ onto $A$ and a total of $d_a$ open disks that are each mapped $1$ to $1$ onto $D_a$. Similarly, inside $D_b$ there is another annulus $B_b$ that is mapped $d_b$ to $1$ onto $A$ and a total of $d_b$ open disks that are each mapped $1$ to $1$ onto $D_b$. Denote these disks by $I_{a,1}, \ldots, I_{a,d_a}$ and $I_{b,1}, \ldots, I_{b,d_b}$. See Figure 3.5.

**Remark 3.3.3** Figure 3.5 is a schematic picture where the curves $\rho$ and $\tau$ seem to be disjoint. This is not necessarily the case but, as the reader may easily check, this does not affect the result.

For each $i = 1, \ldots, d_a$ we have that, since each $I_{a,i}$ is mapped univalently over the union of all of the $I_{a,i}$, the set of points whose orbits remain for all iterations in the union of the $I_{a,i}$ forms a Cantor set on which $f_\lambda$ is conjugate to the one-sided
shift map on \(d_a\) symbols. Similarly, for each \(j = 1, \ldots, d_b\) we have that, since each \(I_{b,j}\) is mapped univalently over the union of all of the \(I_{b,j}\), the set of points whose orbits remain for all iterations in the union of the \(I_{b,j}\) forms a Cantor set on which \(f_\lambda\) is conjugate to the one-sided shift map on \(d_b\) symbols. This follows from standard arguments in complex dynamics [Mil06]. This also produces an uncountable number of point components in \(J\). However, there are many other point components in \(J\), as any point whose orbit eventually lands in the Cantor sets is also in \(J\). There are, however, still other points in \(J\), as we show below.

To understand the complete structure of the Julia set, we show that \(J\) is homeomorphic to a quotient of a subset of the space of one sided sequences of finitely many symbols. Moreover, we show that \(f_\lambda\) on \(J\) is conjugate to a certain quotient of a subshift of finite type on this space. This will prove our main results in the case \(|a|, |b| < 1\) with \(b \neq a^n\) (or \(b = a^n\) with \(d_a > d_b - 1\)).
To begin the construction of the sequence space, we first partition the annulus $A$ into $n$ “rectangles” that are mapped over $A$ by $f_\lambda$.

**Proposition 3.3.4** There is an arc $\xi$ lying in $A$ and having the property that $f_\lambda$ maps $\xi$ to 1 onto a larger arc that properly contains $\xi$ and connects the curve $\rho$ and the circle of radius $s$. Moreover, $\xi$ meets $\partial B$ at exactly one point, namely one of the fixed points in $\partial B$. With the exception of this fixed point, all other points on $\xi$ lie in the Fatou set.

**Proof:** Let $p = p_\lambda$ be one of the repelling fixed points on $\partial B$. Note that $p$ varies analytically with both $\lambda$ and the poles $a$ and $b$. As is well known, there is an invariant external ray in $B$ extending from $p$ to $\infty$. Define the portion of $\xi$ in $B \cap A$ to be the piece of this external ray that lies in $A$.

To define the piece of $\xi$ lying inside $\partial B$, let $U$ be an open set that contains $p$ and meets some portion of $\tau$ and also has the property that the branch of the inverse of $f_\lambda$ that fixes $p$ is well-defined on $U$. Let $f_\lambda^{-1}$ denote this branch of the inverse of $f_\lambda$. Let $w \in \tau \cap U$ and choose any arc in $U$ that connects $w$ to $f_\lambda^{-1}(w)$. Then we let the remainder of the curve $\xi$ be the union of the pullbacks of this arc by $f_\lambda^{-k}$ for all $k \geq 0$. Note that this curve limits on $p$ as $k \to \infty$.

\[ \square \]

We now partition $A$ into $n$ rectangles. Consider the $n$ preimages of $f_\lambda(\xi)$ that lie in $A$. Denote these preimages by $\xi_1, \ldots, \xi_n$ where $\xi_1 = \xi$ and the remaining $\xi_j$'s are arranged counterclockwise around $A$. Let $A_j$ denote the closed region in $A$ that is bounded by $\xi_j$ and $\xi_{j+1}$, so that $A_n$ is bounded by $\xi_n$ and $\xi_1$. By construction, each $A_j$ is mapped 1 to 1 over $A$ except on the boundary arcs $\xi_j$ and $\xi_{j+1}$, which are each mapped 1 to 1 onto $f_\lambda(\xi_1) \supset \xi_1$. 

Now recall that the only points whose orbits remain for all iterations in $\mathcal{A}$ are those points on the simple closed curve $\partial B$. Let $z \in \partial B$. We may attach a symbolic sequence $S(z)$ to $z$ as follows. Consider the $n$ symbols $\alpha_1, \ldots, \alpha_n$. Define $S(z) = (s_0s_1s_2\ldots)$ where each $s_j$ is one of the symbols $\alpha_1, \ldots, \alpha_n$ and $s_j = \alpha_k$ if and only if $f^j_\lambda(z) \in A_k$. Note that there are two sequences attached to $p$, the sequences $(\overline{\alpha_1})$ and $(\overline{\alpha_n})$. Similarly, if $z \in \xi_k \cap \partial B$, then there are also two sequences attached to $z$, namely $(\alpha_{k-1}\overline{\alpha_n})$ and $(\alpha_k\overline{\alpha_1})$. Finally, if $f^j_\lambda(z) \in \xi_k$, then there are again two sequences attached to $z$, namely $(s_0s_1\ldots s_{j-1}\alpha_{k-1}\overline{\alpha_n})$ and $(s_0s_1\ldots s_{j-1}\alpha_k\overline{\alpha_1})$.

Note that if we make the above identifications in the space of all one-sided sequences of the $\alpha_j$'s, then this is precisely the same identifications that are made in coding the itineraries of the map $z \mapsto z^n$ on the unit circle. So this sequence space with these identifications and the usual quotient topology is homeomorphic to the unit circle and the shift map on this space is conjugate to $z \mapsto z^n$.

We now extend this symbolic dynamics to the set of points in the annuli $\mathcal{B}_a$ and $\mathcal{B}_b$ that are mapped to $\partial B$. Recall that $f_\lambda$ takes $\mathcal{B}_a$ (resp. $\mathcal{B}_b$) onto $\mathcal{A}$ as a $d_a$ to 1 (resp. $d_b$ to 1) covering. We let $\nu_{a,1}, \ldots, \nu_{a,d_a}$ be the $d_a$ preimages of $\xi_1$ in $\mathcal{B}_a$ arranged in counterclockwise order beginning at some given preimage. In the same way we let $\nu_{b,1}, \ldots, \nu_{b,d_b}$ be the $d_b$ preimages of $\xi_1$ in $\mathcal{B}_b$ arranged in counterclockwise order beginning at some given preimage. Define the rectangles $B_{a,i}, i = 1, \ldots, d_a$ (resp. $B_{b,j}$ for $j = 1, \ldots, d_b$) to be the region in $\mathcal{B}_a$ (resp. $\mathcal{B}_b$) bounded by $\nu_{a,i}$ and $\nu_{a,i+1}$ (resp. $\nu_{b,j}$ and $\nu_{b,j+1}$) where $B_{d_a}$ (resp. $B_{d_b}$) is bounded by $\nu_{a,d_a}$ and $\nu_{a,1}$ (resp. $\nu_{b,d_b}$ and $\nu_{b,1}$). We introduce $d = d_a + d_b$ new symbols $\beta_{a,1}, \ldots, \beta_{a,d_a}$ and $\beta_{b,1}, \ldots, \beta_{b,d_b}$ and define the itinerary $S(z)$ for a point $z \in \mathcal{B}_a$ (resp. $\mathcal{B}_b$) that is mapped to $\partial B$ as follows. $S(z)$ is a sequence that begins with a single $\beta_{a,k}$ (resp. $\beta_{b,k}$) and then ends in any sequence of $\alpha_j$'s.
From the above, we need to make the following additional identifications in this larger set of sequences:

1. The sequences \((\beta_{a,1}\overline{a_1})\) and \((\beta_{a,d_1}\overline{a_m})\);

2. The sequences \((\beta_{a,k-1}\overline{a_n})\) and \((\beta_{a,k}\overline{a_1})\);

3. The sequences \((\beta_{b,1}\overline{a_1})\) and \((\beta_{b,d_b}\overline{a_m})\);

4. The sequences \((\beta_{b,k-1}\overline{a_n})\) and \((\beta_{b,k}\overline{a_1})\).

Finally, we extend the definition of \(S(z)\) to any point in \(J\) that remains in the union of the \(I_{a,i}\) (resp. \(I_{b,j}\)) by introducing the symbols \(1, \ldots, d\) and defining \(S(z)\) in the usual manner. This defines the itinerary of any point in \(J\) whose orbit:

1. Remains in \(A\) for all iterations;

2. Starts in \(B_a\) or \(B_b\) and then remains for all subsequent iterations in \(A\);

3. Remains in \(I_{a,i} \cup I_{b,j}\) for all iterations.

To extend this definition to all of \(J\), we let \(\Sigma'\) denote the set of sequences involving the symbols \(\alpha_1, \ldots, \alpha_n, \beta_{a,1}, \ldots, \beta_{a,d_a}, \beta_{b,1}, \ldots, \beta_{b,d_b}, 1, \ldots, d\) subject to the following restrictions:

1. Any symbol can follow \(\alpha_j\);

2. The symbol \(\beta_{a,i}\) or \(\beta_{b,j}\) can only be followed by \(\alpha_1, \ldots, \alpha_n\);

3. The symbols \(1, \ldots, d\) can only be followed by \(1, \ldots, d, \beta_{a,1}, \ldots, \beta_{a,d_a}\) or \(\beta_{b,1}, \ldots, \beta_{b,d_b}\).

Let \(\Sigma\) denote the space \(\Sigma'\) where we extend the above identifications to any pair of sequences that ends in a pair of sequences identified earlier. For example, we identify the two sequences \(s_0s_1 \ldots s_n\alpha_{k-1}\overline{a_n}\) and \(s_0s_1 \ldots s_n\alpha_k\overline{a_1}\). We endow \(\Sigma\) with
Figure 3.6: Diagram for the case when $|a| < 1, b = a^n$ and $d_a < d_b - 1$.

The grey annulus around $a$ contains the set $S_a$ and is mapped $d_a$ to 1 onto the trap door $T_b$. The curve $\rho$ is a preimage of the curve $\tau$. The boundary of $\mathcal{D}_b$ and the curve $\tau$ are preimages of the circle of radius $r$. The circle of radius $r$ contains the critical values corresponding to the sets $S_q$ and $S_b$. There are $d_b$ disks inside $\mathcal{D}_b$ and $n - 1$ disks in between $\rho$ and $\tau$ that are preimages of $\mathcal{D}_b$.

the quotient topology. Then, by construction, the Julia set of $f_\lambda$ is homeomorphic to $\Sigma$ and $f_\lambda | J$ is conjugate to the shift map on $\Sigma$. This proves the main results in the case $|a|, |b| < 1$ and $b \neq a^n$ (or $b = a^n$ with $d_a > d_b - 1$).

Figure 3.1 shows an example of the Julia set of $f_\lambda(z) = z^2 + \lambda/((z - a)^4(z - b)^5)$ when $a = 1/2$ and $b = -1/3i$.

For the case $|a| < 1$ with $b = a^n$ and $d_a < d_b - 1$ we need to take into account the following changes. As shown in Section § 3.1, we may assume that $|\lambda|$ is small enough so that the critical values corresponding to the critical points in $S_q$ and $S_b$ lie in $Q$ and the critical values corresponding to the critical points in $S_a$ lie in $T_b$. None of the critical values lie in $\mathcal{T}_a$ or in any of the preimages of this set. Therefore we have that all the preimages of $\mathcal{T}_a$ are closed disks that are mapped univalently onto $\mathcal{T}_a$ by some iterate of $f_\lambda$. 
Consider a circle of radius $r$ centered at the origin that contains the critical values corresponding to the critical points in $S_a$ and $S_b$. There is a preimage of this circle, a curve that we call $\tau$ that surrounds the origin and contains the pole $b$. Notice that the pole $a$ lies outside $\tau$. Consider the annular region between the circle of radius $r$ and $\tau$. Since the critical points inside this annulus $S_b$ are mapped inside the circle of radius $r$ centered at the origin, it follows that there is an open disk about $b$ that is mapped as a $d_b$ to 1 covering (except at $b$) onto the exterior of the circle of radius $r$ (excluding the point at $\infty$). Let $D_b$ denote this region. Then all points inside the curve $\tau$ except for those inside $D_b$ are mapped inside $|z| = r$. Inside $D_b$ there is an annulus $B_b$ that is mapped $d_b$ to 1 to $A$ and a total of $d_b$ open disks that are each mapped 1 to 1 onto $D_b$.

There is also a curve $\rho$ that is a preimage of $\tau$ and contains the pole $a$. When $|\lambda| << 1$ we have that far from the poles the map looks like $z \rightarrow z^n$ and so there are $n$ preimages of $\partial D_b$ distributed around the origin in the annulus between $\tau$ and $\rho$. Since $a$ is mapped to infinity there is another curve that surrounds $a$ and is mapped $d_a$ to 1 onto $\partial D_b$. Then, the annulus between $\tau$ and $\rho$ contains $n - 1$ copies of $D_b$. Then, we have found $n + d_b - 1$ preimages of $T_b$ inside $\Gamma$. The $d_a + 1$ critical points around $a$ are mapped inside $T_b$ and so the other $d_a + 1$ preimages of $T_b$ must contain these critical points. We have a map of degree $d_a + 1$ that takes a set of connectivity $\sigma$ to $T_b$ of connectivity $t = 1$ and there are $N = d_a + 1$ critical points in the domain. By the Riemann-Hurwitz formula we have

$$\sigma - 2 = (d_a + 1)(t - 2) + N.$$

Therefore $\sigma = 2$, that is, the preimage of $T_b$ that contains the set $S_a$ is an annulus that surrounds the pole $a$. See Figure 3.6. We have:
Proposition 3.3.5 Let $|a| < 1$, $b = a^n$ and $d_a < d_b - 1$. Then, when $\lambda$ is sufficiently small there is a preimage of $T_b$ that contains the set $S_a$ and is an annulus that surrounds the pole $a$.

The construction for the symbolic dynamics of this case can be obtained in a similar way as for the case when $b \neq a^n$.

We have proved our theorems for the case when $a_{\text{max}} < 1$.

Figure 3.2 shows the Julia set of $f_\lambda(z) = z^5 + \lambda((z - a)^3(z - b)^7)$ when $a = \frac{3}{4}$ and $b = a^5$.

3.4 Structure of the Julia set for $|a|, |b| > 1$

There is a special kind of symmetry between the cases $0 < a_{\text{max}} < 1$ and $a_{\text{min}} > 1$. In Proposition 3.3.1 we proved that when $|\lambda| << 1$ the trap doors $T_a$ and $T_b$ and the immediate basin of attraction of infinity $B$ were disjoint. This implied that all the preimages of $B$ were disjoint and also implied the existence of uncountably many components accumulating on the boundary of those preimages of $B$. Using the same argument it is easy to see that for $a_{\text{min}} > 1$ and $|\lambda| << 1$ the immediate basin of $q$ and its preimages are disjoint as well.

When $a_{\text{min}} > 1$ there are only $n - 1$ critical points in $Q$ so there must be $d$ preimages of $Q$ somewhere else. By Proposition 3.1.2 these preimages cannot lie outside a circle of radius greater than $a_{\text{max}}$. We shall prove that $B$ is infinitely connected and the Julia set is made up of a countable number of simple closed curves and uncountably many point components that accumulate on these curves.

By Corollary 3.1.6 the critical values corresponding to $S_a$ and $S_b$ are in $B$, so we next prove that the sets $S_a$ and $S_b$ themselves lie in $B$.

We consider first the case when $b \neq a^n$. 

**Proposition 3.4.1** When $a_{\min} > 1$ and $|\lambda|$ is sufficiently small the critical points surrounding the poles $a$ and $b$ are contained in $B$ and $B$ is not simply connected.

**Proof:** Fix $n \geq 2$, $d_a, d_b \geq 1$ and the poles $a$ and $b$ so that $a_{\min} = \min\{|a|, |b|\} > 1$. Let $\mu$ be a simple closed curve that lies outside $\mathcal{Q}$ and separates the poles $a$ and $b$ from the points $a^n$ and $b^n$. Let $\lambda$ be sufficiently small so that:

1. $f_\lambda$ has an attracting fixed point $q$ near the origin;

2. The $n - 1$ critical points surrounding $q$ are in $\mathcal{Q}$;

3. There is a preimage of the curve $\mu$, another simple closed curve that we call $\eta$, such that $\eta$ lies outside $\mathcal{Q}$ and the poles $a$ and $b$ are in between $\eta$ and $\mu$.

For $\lambda$ small enough the set $S_a$ lies close to $a$ and the set $S_b$ lies close to $b$. Thus we have a function of degree $n + d$ that maps the exterior of $\eta$ (connectivity $\sigma$) to the exterior of $\mu$ (connectivity $t = 1$) and there are $N = 2d + n - 1$ critical points in the domain $(S_a, S_b, a, b$ and $\infty)$. By the Riemann-Hurwitz formula

$$\sigma - 2 = (n + d)(t - 2) + N.$$ 

We get $\sigma = d + 1$ and so there are $d$ holes in the region bounded by $\eta$ and $\mu$. These holes contain preimages of the interior of $\mu$. It follows that these holes contain the $d$ other preimages of $\mathcal{Q}$ so $B$ is not simply connected.

\[\Box\]

As before (see Propositions 3.3.1 and 3.3.2), it follows that there must be other components of the Julia set (uncountably many) accumulating on the boundaries of the preimages of $\mathcal{Q}$. However, the symbolic dynamics on the Julia set is somewhat different.
The regions $\mathcal{A}$ and the $I_j$'s when $a_{\min} > 1$. The annulus $\mathcal{A}$ is bounded by the curve $\gamma$ and the circle of radius $r$ centered at the origin.

To begin the construction of the sequence space we first choose $|\lambda|$ sufficiently small so that

1. All critical points in $S_a$ and $S_b$ lie in the region between the curves $\mu$ and $\eta$;

2. All critical values that correspond to $S_a$ and $S_b$ lie outside the curve $\mu$.

There are $d$ preimages of the disk bounded by $\mu$ that we call $I_1, \ldots, I_d$ and lie inside the region between $\eta$ and $\mu$. Each $I_j$ is mapped 1 to 1 to the disk bounded by $\mu$. The set of points whose orbits remain for all iterations in the union of the $I_j$ forms a Cantor set on which $f_\lambda$ is conjugate to the one-sided shift map on $d$ symbols. This produces an uncountable number of point components in $J$. However, as in the case $a_{\max} < 1$, there are many other point components in $J$.

Consider a circle of radius $r < 1$ that lies completely inside $\mathcal{Q}$ and contains all the critical points in $S_q$. Call $\mathcal{A}$ the annulus between this circle and the curve $\eta$. Then,
\( \partial Q \subset A \) and notice that each \( I_j \) contains a copy of \( \overline{Q} \) and a copy of each one of the \( I_j \)'s. See Figure 3.7.

**Remark 3.4.2** Figure 3.7 is a schematic picture where the curves \( \eta \) and \( \mu \) seem to be disjoint. This is not necessarily the case but, as the reader may easily check, this does not affect the result.

As we did in the case \( a_{\max} < 1 \) (Proposition 3.3.4) we can find an arc \( \xi \) lying in \( A \) and having the property that \( f_\lambda \) maps \( \xi \) to 1 onto a larger arc that properly contains \( \xi \) and connects the circle of radius \( r \) and \( \eta \). Moreover, \( \xi \) meets \( \partial Q \) at exactly one point, namely one of the fixed points in \( \partial Q \). With the exception of this fixed point, all other points on \( \xi \) lie in the Fatou set.

Similarly, we now partition \( A \) into \( n \) rectangles using the \( n \) preimages of \( f_\lambda(\xi) \) that lie in \( A \). Again denote these preimages by \( \xi_1, \ldots, \xi_n \) where \( \xi_1 = \xi \) and the remaining \( \xi_j \)'s are arranged counterclockwise around \( A \). Let \( A_j \) denote the closed region in \( A \) that is bounded by \( \xi_j \) and \( \xi_{j+1} \), so that \( A_n \) is bounded by \( \xi_n \) and \( \xi_1 \). By construction, each \( A_j \) is mapped 1 to 1 over \( A \) except on the boundary arcs \( \xi_j \) and \( \xi_{j+1} \), which are each mapped 1 to 1 onto \( f_\lambda(\xi_1) \supset \xi_1 \). As before the only points whose orbits remain for all iterations in \( A \) are those points on the simple closed curve \( \partial Q \). Let \( z \in \partial Q \). We may attach a symbol sequence \( S(z) \) to \( z \) as follows. Consider the \( n \) symbols \( \alpha_1, \ldots, \alpha_n \). Define \( S(z) = (s_0s_1s_2 \ldots) \) where each \( s_j \) is one of the symbols \( \alpha_1, \ldots, \alpha_n \) and \( s_j = \alpha_k \) if and only if \( f_\lambda^j(z) \in A_k \). We use the same identifications for points in \( \partial Q \) as we used for points in \( \partial B \) when \( a_{\max} < 1 \). Then, \( f_\lambda|\partial Q \) is conjugate to \( z \to z^n \) on the unit circle.

Finally, we extend the definition of \( S(z) \) to any point in \( J \) that remains in the union of the \( I_j \)'s by introducing the symbols \( 1, \ldots, d \) and defining \( S(z) \) in the usual manner. We identify the sequences of the form \((j, \alpha_1)\) and \((j, \alpha_n)\) as well as \((j, \alpha_k, \alpha_n)\) and \((j, \alpha_k, \alpha_1)\).
Let $\Sigma'$ denote the set of sequences with symbols $\alpha_1, \ldots, \alpha_n, 1, \ldots, d$. Let $\Sigma$ denote the space $\Sigma'$ with all of the identifications described above and endow $\Sigma$ with the quotient topology. Then, by construction, the Julia set of $f_\lambda$ is homeomorphic to $\Sigma$ and $f_\lambda | J$ is conjugate to the full shift map on $\Sigma$.

Figure 3.3 shows an example of the Julia set of $f_\lambda(z) = z^3 + \lambda/((z - a)^3(z - b)^4)$ when $a$ and $b$ are outside the unit circle.

When $b = a^n$ we do the following. Let $\lambda$ be small enough and consider a circle $\mu$ around 0 such that:

1. the radius of $\mu$ is larger than $|b|$;
2. no critical value lies on $\mu$;
3. the critical values corresponding to the critical points in $S_b$ lie outside $\mu$ (and around $b^n$).

There is a curve $\eta$ that is a preimage of $\mu$ such that, in the region between $\eta$ and $\mu$ we find the pole $b$ and the critical values corresponding to the critical points in $S_a$ but the pole $a$ is inside $\eta$. Then, the function $f_\lambda$ is of degree $n + d_b$ in the region outside $\eta$. The Riemann-Hurwitz formula applied to the regions outside $\eta$ and outside $\mu$ gives $d_b$ holes in between $\eta$ and $\mu$. These holes contain preimages of the interior of $\mu$ and so, they contain $d_b$ preimages of the interior of $\Gamma$.

Remark 3.1.5 implies that when $\lambda$ is sufficiently small there are also $d_a$ preimages of the interior of $\Gamma$ symmetrically distributed around the pole $a$. Since the critical points in $S_a$ are mapped close to $a^n$ the Riemann-Hurwitz formula implies that these preimages of the interior of $\Gamma$ are disjoint. Then, we have $d = d_a + d_b$ preimages of $Q$ outside the curve $\Gamma$ that are each mapped 1 to 1 onto $Q$.

As in the previous case we have that the curve $\Gamma$ and its preimages form a countable set of simple closed curves in the Julia set of $f_\lambda$ and the other components of $J$ are
points. The symbolic dynamics for this case are essentially the same as for the case when \( b \neq a^n \).

We have proved our main theorems for the case \(|a|, |b| > 1\).

### 3.5 Structure of the Julia set for \(|a| < 1\) and \(|b| > 1\)

Assume that \(|a| < 1\) and \(|b| > 1\). Let \( \lambda \) be sufficiently small and let \( \rho \) be a circle around the origin that is contained in \( Q \) and contains the point \( a^n \) as well as the critical values corresponding to the critical points in \( S_a \) and \( S_q \). There is a preimage of \( \rho \), a curve \( \tau \) that is also contained in \( Q \) and contains the pole \( a \).

We define another circle \( \mu \) contained in \( B \) that separates the pole \( b \) from \( b^n \). The circle \( \mu \) has a preimage, a curve we call \( \eta \) that is contained in \( B \) and such that the pole \( b \) lies outside \( \eta \).

Call \( \mathcal{A} \) the annulus defined by the curves \( \tau \) and \( \eta \). The map \( f_\lambda \) is an \( n \) covering map in \( \mathcal{A} \). As we did in the previous cases we can partition \( \mathcal{A} \) in \( n \) rectangles. The points that for all iterations remain in \( \mathcal{A} \) belong to the only component of \( J \) that surrounds the origin, that is \( \Gamma \), that separates \( Q \) from \( B \). The map \( f_\lambda \) restricted to \( \Gamma \) is conjugate to the map \( z \to z^n \) on the unit circle. See Figure 3.8.

As defined in the previous sections there is a disk \( D_a \) around \( a \) that is mapped outside the circle \( \rho \) in a \( d_a \) to 1 fashion. There are also disks \( I_i \) for \( i = 1, \ldots, d_b \) around \( b \) that are mapped to the interior of the circle \( \mu \) in a 1 to 1 fashion. So, each disk contains preimages of all the others. It follows that the Julia set of \( f_\lambda \) consists of a countable collection of simple closed curves and uncountably many point components that accumulate on those curves. Every Fatou component is infinitely connected.

Continuing as in the previous cases it is possible to prove that the Julia set of \( f_\lambda \) is homeomorphic to a quotient of a subshift of finite type on finitely many symbols.
Figure 3.8: Diagram for the case when $|a| < 1$ and $|b| > 1$.

The regions $\mathcal{A}$, $\mathcal{D}_a$ and $I_j$'s with $j = 1, \ldots, d_b$ when $a_{\min} < 1$ and $a_{\max} > 1$. The annulus $\mathcal{A}$ is bounded by the curve $\tau$ and the curve $\eta$.

This finishes the proof of our main theorems.

Figure 3.4 shows an example of the Julia set of $f_\lambda(z) = z^2 + \lambda/(z - a)^5(z - b)^4$ when $|a| < 1$ and $|b| > 1$. 
Chapter 4

Singular Perturbations in the McMullen Domain

In this chapter we prove Theorems 1.0.6 and 1.0.7. Then, the function $f_\lambda$ from Eq. (1.1) can be written as:

$$f_\lambda(z) = z^n + \frac{\lambda}{z^h(z - a)^d}$$

where $1/n + 1/h < 1$, $d \geq 1$, $b = 0$ and $|a| > 1$. To simplify the notation we have introduced $h = d_b$ and $d = d_a$. The condition on $n$ and $h$ is equivalent to the requirement that $n$ and $h$ be greater than or equal to 2 but not simultaneously equal to 2.

4.1 Preliminaries

A straightforward calculation shows that the critical points of $f_\lambda$ satisfy

$$nz^{2h+n-1}(z - a)^{2d} = \lambda(hz^{h-1}(z - a)^d + d(z - a)^{d-1}z^h).$$
Notice that there is a critical point of order \( h - 1 \) at the origin and a critical point of order \( d - 1 \) at the pole \( a \). If we remove these solutions from the above equation and let \( c = c_\lambda \) be any other critical point of \( f_\lambda \), then \( c \) must satisfy

\[
n c^{n+h}(c - a)^{d+1} = \lambda (h(c - a) + dc). \tag{4.1}
\]

When \( \lambda = 0 \) this equation has \( n + h + d + 1 \) roots, the origin with multiplicity \( n + h \) and the pole \( a \) with multiplicity \( d + 1 \). By continuity, for small enough \( |\lambda| \), these roots become simple zeros of \( f_\lambda \) that lie in the plane approximately symmetrically distributed around the origin and the pole \( a \). Indeed, when \( \lambda \) is sufficiently small, \( n + h \) of the critical points of \( f_\lambda \) are grouped around the origin while \( d + 1 \) of the critical points are grouped around the pole \( a \).

The critical value corresponding to \( c \) is given by

\[
v = f_\lambda(c) = c^n + \frac{\lambda}{c^h(c - a)^d}
\]

and, using Eq. (4.1), we have

\[
v = c^n + \frac{nc^n(c - a)}{h(c - a) + dc}. \tag{4.2}
\]

From Eq. (4.2) it follows that, if \( c \to 0 \), then \( v \to 0 \). Similarly, if \( c \to a \), then \( v \to a^n \).

Let \( S_a = \{d + 1 \text{ critical points around } a\} \) and \( S_0 = \{n + h \text{ critical points around } 0\} \).

We have:

**Proposition 4.1.1** *(Location of the critical values)* When \( |\lambda| \) tends to zero the critical values corresponding to the critical points in \( S_0 \) tend to the origin and the critical values corresponding to the critical points in \( S_a \) tend to \( a^n \).
When $\lambda = 0$, $f_\lambda$ is hyperbolic on the unit circle. Thus, for small enough $\lambda \neq 0$ there exists a simple closed curve $\Gamma$ close to the unit circle such that the map $f_\lambda$ acts on $\Gamma$ as $z \rightarrow z^n$. The curve $\Gamma$ is a boundary component of $B$.

The following proposition gives a bound on the location of the Julia set.

**Proposition 4.1.2** Suppose $1/n + 1/h < 1$, $d \geq 1$ and $|a| > 1$. Let $|z| = r$. If $r > |a|$ then, for sufficiently small $\lambda$, $z \in B$.

**Proof:** Fix $n, h, d$ and $a$ such that $1/n + 1/d < 1$, $d \geq 1$ and $|a| > 1$. We write $|z - a| \geq |z| - |a| = r - |a| > 0$. Then we have

$$|f_\lambda(z)| \geq |z|^n - \frac{|\lambda|}{|z|^h |z - a|^d} \geq r^n - \frac{|\lambda|}{r^h (r - |a|)^d}.$$ 

Let $|\lambda| < r^h (r - |a|)^d(r^n - r)$. Then

$$|f_\lambda(z)| > r^n - \frac{r^h (r - |a|)^d(r^n - r)}{r^h (r - |a|)^d} = r^n - r^n + r = r.$$ 

Hence $|f_\lambda(z)| > |z|$ and the orbit of $z$ converges to $\infty$ so that $z \in B$.

\[\square\]

Now we give some estimates for the size of the trap door $T$ that surrounds the origin and the approximate position of the critical points in $S_0$ and their corresponding critical values.

**Proposition 4.1.3** (*Estimates for T and S₀*) When $|\lambda| \ll 1$ we have that:

1. the critical points in $S_0$ are close to the $(n + h)$-roots of $\lambda h / (n(-a)^d)$ and,

2. there is a simple closed curve $\gamma$ around the origin that is a preimage of $\Gamma$ and lies close to a circle of radius $(|\lambda|/|a|^d)^{1/h}$.
We will show later that the trap door as well as all components of the Fatou set are infinitely connected. All the punctures and holes in $T$ are bounded inside $\gamma$, so the curve $\gamma$ is the outer boundary of the trap door $T$.

**Proof:** Fix $n, h, d$ and $a$ with $1/n + 1/h < 1$, $d \geq 1$ and $|a| > 1$. For the first part let $z = (\lambda h/(n(-a)^d))^{1/(n+h)}$ and replace $c$ by $z$ in Eq. (4.1) to get

$$\frac{h\lambda}{(-a)^d}((\frac{\lambda h}{n(-a)^d})^{1/(n+h)} - a)^{d+1} = \lambda(h((\frac{\lambda h}{n(-a)^d})^{1/(n+h)} - a) + d((\frac{\lambda h}{n(-a)^d})^{1/(n+h)}).$$

Cancel $\lambda$ on both sides and then let $\lambda$ go to zero. It is easy to see that, when $\lambda \to 0$, then $z \to 0$ and so $(z - a) \to (-a)$. Thus, when $\lambda$ is sufficiently small, we can simplify both sides of the above equation to get an identity and the result follows.

For part 2, when $|\lambda|$ is sufficiently small, the curve $\Gamma$ is close to the unit circle. Since $\gamma$ is a preimage of $\Gamma$ we then have that the curve $\gamma$ is mapped close to the unit circle. Let $z = (|\lambda|/|a|^d)^{1/h} e^{i\theta}$ and let $\lambda = |\lambda|e^{i\phi}$. Then apply $f_\lambda$ to $z$, i.e.,

$$f_\lambda(z) = (|\lambda|/|a|^d)^{1/h} e^{i\theta} + \frac{|\lambda|e^{i\phi}}{(|\lambda|/|a|^d)^{1/h} e^{i\theta}((|\lambda|/|a|^d)^{1/h} e^{i\theta} - a)^d}.$$

Cancel $|\lambda|$ in the second term and then let $\lambda \to 0$. We get $|f_\lambda(z)| \to |e^{i(\phi-h\theta)}| = 1$ as we wanted to show.

Part 2 of Proposition 4.1.3 implies that when $|\lambda| << 1$ there are $h$ preimages of the exterior of the curve $\Gamma$ inside the curve $\gamma$. This in turn implies that outside $\Gamma$ the map $f_\lambda$ has degree $n + d$. Also, the map is of degree $n$ in $B$ so that there are $d$ preimages of the interior of $\Gamma$ outside $\Gamma$ and near the pole $a$. Therefore, the map is of degree $n + h$ in the annulus between $\gamma$ and $\Gamma$.

To show the existence of a Cantor set of simple closed curves that surrounds the
origin we show (as in the case when \( d = 0 \)) that when \( \lambda \) is sufficiently small the
critical values corresponding to the critical points in \( S_0 \) lie inside the trap door \( T \).
The following proposition implies this result.

**Proposition 4.1.4** When \( |\lambda| \to 0 \) the critical values \( v \) that correspond to the critical
points in \( S_0 \) are such that \( |v| \to 0 \) faster than \( |\lambda|^{1/h} \).

**Proof:** Let \( z = (\frac{\lambda h}{n(-a)^d})^{1/(n+h)} \) and compute \( f_\lambda(z) \). We have,

\[
v = f_\lambda(c) \approx f_\lambda(z) = \left( \frac{\lambda h}{n(-a)^d} \right)^{n/(n+h)} + \frac{\lambda}{\left( \frac{\lambda h}{n(-a)^d} \right)^{1/(n+h)} - a^d}
\]

that is, \( v \approx \lambda^{n/(n+h)} K(\lambda) \) where \( K(\lambda) \) tends to a finite value when \( \lambda \to 0 \). Since
\( 1/n + 1/h < 1 \) is equivalent to \( n/(n + h) > 1/h \), we get the result.

\( \square \)

**Corollary 4.1.5** When \( |\lambda| \) is sufficiently small the critical values corresponding to
the critical points in \( S_0 \) lie in the trap door \( T \) and the critical values corresponding to
the critical points in \( S_a \) lie in \( B \).

### 4.2 The structure of the Julia set

Let \( |\lambda| \ll 1 \) so that we can define a disk \( \tau \) that is bounded by the curve \( \gamma \) (the
outer boundary of the trap door \( T \)) and such that \( \tau \) contains the critical values
 corresponding to the critical points in \( S_0 \). The fact that \( \tau \neq T \) will be clear after we
prove the next two propositions.

**Proposition 4.2.1** There is a preimage of \( \tau \) that is an annulus \( A \) that is contained
in between \( \gamma \) and \( \Gamma \). The annulus \( A \) contains the set \( S_0 \). All the preimages of \( A \) are
annuli and, by construction, the union of all these annuli forms the complement of a Cantor set of simple closed curves. Each one of these curves is in \( J \).

We remark that the existence of the Cantor set of circles for the family \( z^n + \lambda/z^h \) with \( 1/n + 1/h < 1 \) is due to McMullen [McM88].

**Proof:** Restricted to the region between \( T \) and \( \Gamma \) the map \( f_\lambda \) is of degree \( n + h \). Let \( \sigma \) be the connectivity of the preimage of \( \tau \) that lies in this region. The connectivity of \( \tau \) is \( t = 1 \). There are \( n + h \) critical points in the preimage of \( \tau \). Then, applying the Riemann-Hurwitz formula, we get

\[
\sigma - 2 = (n + h)(t - 2) + n + h.
\]

This implies that \( \sigma = 2 \). Thus, the preimage of the disk \( \tau \) has two boundary components and it is therefore an annulus. Taking preimages of \( \mathcal{A} \) we get the result.

\[\square\]

Figure 4.1 shows the first step in the construction of the Cantor set of simple closed curves that surrounds the origin. Notice that the above proposition proves that when \( |\lambda| \ll 1 \), \( T \) and \( B \) are disjoint.

Now we prove that there are preimages of the Cantor set of simple closed curves outside \( \Gamma \).

**Proposition 4.2.2** For sufficiently small \( \lambda \) the critical points in \( S_a \) are contained in \( B \) and \( B \) is not simply connected.

**Proof:** Let \( \eta \) be a simple closed curve that separates the pole \( a \) from the point \( a^n \) and let \( \lambda \) be sufficiently small so that outside of \( \eta \) we have all the critical values corresponding to the critical points near \( a \). There is a curve \( \mu \) that is a preimage of
Figure 4.1: Diagram for the construction of the Cantor set of circles.

The gray disk is $\tau$, is bounded by $\gamma$ and contains all the critical values corresponding to the critical points in $S_0$. The curve $\gamma$ is the outer boundary component of the trap door $T$. The grey annulus $A$ is the preimage of the disk $\tau$ and contains the $n + h$ critical points in $S_0$.

$\eta$ and such that the pole $a$ and the set of critical points around $a$ lie in the region bounded by $\eta$ and $\mu$.

Then we have a function of degree $n + d$ that maps the exterior of $\mu$ (connectivity $\sigma$) to the exterior of $\eta$ (connectivity $t = 1$) and there are $N = 2d + n - 1$ critical points in the domain $(S_a, a$ and $\infty)$. By the Riemann-Hurwitz formula, we have

$$\sigma - 2 = (n + d)(t - 2) + N,$$

so we get $\sigma = d + 1$ and thus there are $d$ holes in the region bounded by $\eta$ and $\mu$. These holes contain preimages of the interior of $\eta$. It follows that these holes contain the other $d$ preimages of the interior of $\Gamma$ and thus, $B$ is not simply connected.
The curve $\eta$ separates the pole $a$ from $a^n$. The curve $\mu$ is a preimage of $\eta$. The disks $I_i$ for $i = 1, \ldots, d$ are mapped 1 to 1 onto the region bounded by $\eta$. The curve $\Gamma$ is a boundary component of $B$ and it bounds the invariant Cantor set of simple closed curves that surrounds the origin. The gray disks inside the $I_i's$ contain preimages of the Cantor set of circles that surround the origin.

The above proposition shows the existence of preimages of the Cantor sets of circles that surround the origin in the exterior of $\Gamma$. Taking preimages of the exterior of the curve $\mu$ we find countably many Cantor sets of simple closed curves in the Julia set of $f_\lambda$ that accumulate on the exterior of $\Gamma$. Since the trap door contains a preimage of the exterior of $\Gamma$ we get another set of Cantor sets of circles inside the disk $\tau$ and then it follows that every preimage of $T$ contains countably many Cantor sets of circles. We see that every component of the Fatou set is infinitely connected. Part of our main results has been proven. There are, however, other components of the Julia set. Every one of the Cantor sets of circles described above is a preimage of the one that surrounds the origin, so those curves cannot contain periodic points. Repelling periodic points form a countable dense subset of the Julia set $J$ of $f_\lambda$ and so there must be other components of the Julia set accumulating on every one of the curves in the Cantor sets of circles. We show next that these other components of $J$ are points.
The $d$ holes found in Proposition 4.2.2 contain preimages of the interior of $\eta$. We denote these disks by $I_1, I_2, ..., I_d$. It follows that each of these disks is mapped 1 to 1 over the union of all of them. Thus, the set of points whose orbits remain for all iterations in the union of the $I_j$ form a Cantor set on which $f_\lambda$ is conjugate to the one-sided shift map on $d$ symbols. This produces an uncountable number of point components in $J$. However, there are many other point components in $J$, as any point whose orbit eventually lands in this Cantor set of points is also in $J$. There are, however, still other points in $J$, as we show below.

To understand the complete structure of the Julia set, we show that $J$ is homeomorphic to a quotient of a subset of the space of one sided sequences on $n + h + d$ symbols. Moreover, we show that $f_\lambda$ on $J$ is conjugate to a certain quotient of a subshift of finite type on this space. Since this is true for any $|a| > 1$ and $\lambda$ sufficiently small, this will complete the proof of our main results. In order to describe the dynamics on the whole Julia set we do the following.

Consider the curve $\eta$ that is completely contained in $B$ and separates $a$ from $a^n$. Notice that every point outside $\eta$ is in the immediate basin of attraction of $\infty$. As before, let $\mu$ be the preimage of $\eta$ that lies in $B$ and is mapped $n$ to 1 onto $\eta$. To construct the symbolic dynamics we need to consider other preimages of $\eta$. Let $s$ be the preimage of $\eta$ that surrounds the origin and is contained in the trap door $T$. The curve $s$ is mapped $h$ to 1 onto $\eta$. Fix $a$ with $|a| > 1$ then by Proposition 4.1.4 we can take $|\lambda|$ sufficiently small so that the forward images of the critical values corresponding to the critical points in $S_0$ are outside the curve $\eta$. Then, the region bounded by the curve $s$ contains the critical values corresponding to the critical points in $S_0$. The annulus between $s$ and $\eta$ is denoted by $\mathring{A}$.

Recall that there is an annulus $A$ that contains the $n + h$ critical points in $S_0$ and lies between the curves $\gamma$ (exterior boundary of $T$) and $\Gamma$ (interior boundary of $B$).
Figure 4.3: Diagram for the symbolic dynamics on the Julia set.

Definition of the annuli $A_1$ and $A_2$. The annulus $A_1$ is bounded by the curves $\mu$ and $r_1$ and the annulus $A_2$ is bounded by the curves $r_2$ and $s$. The curve $\eta$ separates the pole $a$ from $a^n$. The curves $\mu$ and $s$ are preimages of $\eta$. The disks $I_i$ with $i = 1, \ldots, d$ are mapped 1 to 1 to the interior of $\eta$.

The two preimages of $s$ inside $A$ are disjoint (since the critical points are mapped inside of $s$). We denote by $r_1$ the preimage of $s$ (a simple closed curve) that lies inside $A$ that surrounds the origin and such that all the critical points in $S_0$ lie inside $r_1$. That is, $r_1$ is mapped $n$ to 1 onto $s$. Likewise, let $r_2$ denote the other preimage of $s$ (a simple closed curve) that lies in $A$ and surrounds the origin and such that all the critical points in $S_0$ lie outside $r_2$. That is, $r_2$ is mapped $h$ to 1 onto $s$.

Let $A_1$ denote the annulus between the curves $r_1$ and $\mu$ and let $A_2$ denote the annulus between the curves $r_2$ and $s$. The annulus $A_1$ is mapped as an $n$ to 1 covering onto $\tilde{A}$ and so $A_1$ covers $A_1, A_2$, and the disks $I_i$ for $i = 1, \ldots, d$. That is, the image of $A_1$ contains the Julia set of $f_\lambda$. Similarly, the image of $A_2$ has this property; that is, $A_2$ is mapped as an $h$ to 1 converging onto $\tilde{A}$. See Figure 4.3.
To begin the construction of the sequence space, we first partition the annulus $A_1$ into $n$ "rectangles" that are mapped over $\tilde{A}$ by $f_\lambda$.

**Proposition 4.2.3** There is an arc $\xi$ lying in $A_1$ and having the property that $f_\lambda$ maps $\xi$ 1 to 1 onto a larger arc that properly contains $\xi$ and connects the curve $\mu$ and the curve $r_1$. Moreover, $\xi$ meets $\Gamma$ at one of the repelling fixed points on $\Gamma$.

**Proof:** Let $p = p_\lambda$ be one of the repelling fixed points on $\Gamma$. Note that $p$ varies analytically with both $\lambda$ and $a$. As is well known, there is an invariant external ray in $B$ extending from $p$ to $\infty$. Define the portion of $\xi$ in $B \cap A_1$ to be the piece of this external ray that lies in $A_1$.

To define the piece of $\xi$ lying inside $\Gamma$, let $U$ be an open set that contains $p$ and meets some portion of $r_1$ and also has the property that the branch of the inverse of $f_\lambda$ that fixes $p$ is well-defined on $U$. Let $f_\lambda^{-1}$ denote this branch of the inverse of $f_\lambda$. Let $w \in r_1 \cap U$ and choose any arc in $U$ that connects $w$ to $f_\lambda^{-1}(w)$. Then we let the remainder of the curve $\xi$ be the union of the pullbacks of this arc by $f_\lambda^{-k}$ for all $k \geq 0$. Note that this curve limits on $p$ as $k \to \infty$.

$\square$

We now partition $A_1$ into $n$ rectangles. Consider the $n$ preimages of $f_\lambda(\xi)$ that lie in $A_1$. Denote these preimages by $\xi_1, \ldots, \xi_n$ where $\xi_1 = \xi$ and the remaining $\xi_j$'s are arranged counterclockwise around $A_1$. Let $R_j^1$ denote the closed region in $A_1$ that is bounded by $\xi_j$ and $\xi_{j+1}$, so that $R_n^1$ is bounded by $\xi_n$ and $\xi_1$. By construction, each $R_j^1$ is mapped 1 to 1 over $A_1$ except on the boundary arcs $\xi_j$ and $\xi_{j+1}$, which are each mapped 1 to 1 onto $f_\lambda(\xi_j) \supset \xi_j$.

Now recall that the only points whose orbits remain for all iterations in $A_1$ are those points on the simple closed curve $\Gamma$. Let $z \in \Gamma$. We may attach a symbolic sequence $S(z)$ to $z$ as follows. Consider the $n$ symbols $\alpha_1, \ldots, \alpha_n$. Define $S(z) =$
\((s_0s_1s_2\ldots)\) where each \(s_j\) is one of the symbols \(\alpha_1, \ldots, \alpha_n\) and \(s_j = \alpha_k\) if and only if \(f^j_\lambda(z) \in R_k\). Note that there are two sequences attached to \(p\), the sequences \((\alpha_1)\) and \((\alpha_n)\). Similarly, if \(z \in \xi_k \cap \Gamma\), then there are also two sequences attached to \(z\), namely \((\alpha_{k-1}\alpha_n)\) and \((\alpha_k\alpha_1)\). Finally, if \(f^j_\lambda(z) \in \xi_k\), then there are again two sequences attached to \(z\), namely \((s_0s_1\ldots s_{j-1}\alpha_{k-1}\alpha_n)\) and \((s_0s_1\ldots s_{j-1}\alpha_k\alpha_1)\).

Note that if we make the above identifications in the space of all one-sided sequences of the \(\alpha_j\)'s, then this is precisely the same identifications that are made in coding the itineraries of the map \(z \mapsto z^n\) on the unit circle. So this sequence space with these identifications and the usual quotient topology is homeomorphic to the unit circle and the shift map on this space is conjugate to \(z \mapsto z^n\).

Now we partition the annulus \(A_2\) into \(h\) “rectangles” that are mapped over \(\tilde{A}\) by \(f_\lambda\). Since \(f_\lambda|A_2\) covers itself we see that there is a simple closed curve inside \(A_2\) that is fixed by \(f_\lambda\). We denote this curve by \(\Gamma'\) and note that \(\Gamma'\) is mapped by \(f_\lambda\) as a \(d\) to 1 covering onto itself.

**Proposition 4.2.4** There is an arc \(\xi'\) lying in \(A_2\) and having the property that \(f_\lambda\) maps \(\xi'\) 1 to 1 onto a larger arc that properly contains \(\xi'\) and connects the curve \(s\) and the curve \(r_2\). Moreover, \(\xi'\) meets \(\Gamma'\) at one of the repelling fixed points on \(\Gamma'\).

**Proof:** Let \(p' = p'_\lambda\) be one of the repelling fixed points on \(\Gamma'\). Note that \(p'\) varies analytically with both \(\lambda\) and \(a\). Consider a small neighborhood \(V\) of \(p'\) completely contained in \(A_2\). Since \(p'\) is repelling we have that there is a forward image of \(V\), that we denote \(V'\), such that \(V'\) intersects \(s\) and \(r_2\) and contains \(V\). Consider two points in these intersections and their corresponding preimages in \(V\). Join each one of these points to \(p'\) with simple arcs. Then, pull these arcs back towards \(p'\). This curve is the arc \(\xi'\).

\[\Box\]
We now partition $A_2$ into $h$ rectangles. Consider the $h$ preimages of $f_\lambda(\xi')$ that lie in $A_2$. Denote these preimages by $\xi_1', \ldots, \xi_h'$ where $\xi_1' = \xi'$ and the remaining $\xi_j'$’s are arranged counterclockwise around $A_2$. Let $R_j^2$ denote the closed region in $A_2$ that is bounded by $\xi_j'$ and $\xi_{j+1}'$, so that $R_h^2$ is bounded by $\xi_h'$ and $\xi_1'$. By construction, each $R_j^2$ is mapped 1 to 1 over $A_2$ except on the boundary arcs $\xi_j'$ and $\xi_{j+1}'$, which are each mapped 1 to 1 onto $f_\lambda(\xi_1') \supset \xi_1'$.

As before, the only points whose orbits remain for all iterations in $A_2$ are those points on the simple closed curve $\Gamma'$. Let $z \in \Gamma'$. We may attach a symbolic sequence $S(z)$ to $z$ as follows. Consider the $h$ symbols $\beta_1, \ldots, \beta_h$. Define $S(z) = (s_0 s_1 s_2 \ldots)$ where each $s_j$ is one of the symbols $\beta_1, \ldots, \beta_h$ and $s_j = \beta_k$ if and only if $f_\lambda^j(z) \in R_k^2$. Note that there are two sequences attached to $p'$, the sequences $(\overline{\beta_1})$ and $(\overline{\beta_h})$. Similarly, if $z \in \xi_k' \cap \Gamma'$, then there are also two sequences attached to $z$, namely $(\beta_k \beta_{k-1})$ and $(\beta_k \overline{\beta_1})$. Finally, if $f_\lambda^j(z) \in \xi_k'$, then there are again two sequences attached to $z$, namely $(s_0 s_1 \ldots s_{j-1} \beta_k \beta_{k-1})$ and $(s_0 s_1 \ldots s_{j-1} \beta_k \overline{\beta_1})$.

In this case we have that if we make the above identifications in the space of all one-sided sequences of the $\beta_j$’s, then this is precisely the same identifications that are made in coding the itineraries of the map $z \mapsto z^h$ on the unit circle. So this sequence space with these identifications and the usual quotient topology is homeomorphic to the unit circle and the shift map on this space is conjugate to $z \mapsto z^h$.

We may thus attach a symbol sequence $S(z)$ to any point in $J$ as follows. Let $S(z) = (s_0, s_1, s_2, \ldots, s_j, \ldots)$. Let $\alpha = \{\alpha_i$ with $i = 1, \ldots, n\}$ be a set of $n$ symbols that are to be attached to any point in the Julia set that at some iterate lands on one of the $n$ subrectangles inside $A_1$; and let $\beta = \{\beta_i$ with $i = 1, \ldots, h\}$ be a set of $h$ symbols that are to be attached to any point in the Julia set that at some iterate lands on one of the $h$ subrectangles inside $A_2$. Finally, let $D = \{1, 2, \ldots, d\}$ be a set of $d$ symbols that are to be attached to points that at some iterate land on one of the disks $I_i$ with
i = 1, ..., d. We let \( s_k \in \alpha, \beta \) or \( D \) if the \( k \)-th iterate of \( z \) belongs to \( A_1, A_2 \) or \( I_i \) for \( i = 1, ..., d \), respectively.

The points in the invariant Cantor set of circles that surrounds the origin have itineraries that consists of sequences of \( \alpha_i \)'s and \( \beta_i \)'s. For points on these curves we need to extend the identifications mentioned above on the space of sequences.

Every point in a preimage of the Cantor set of circles that surrounds the origin that is inside one of the disks \( I_i \) with \( i = 1, ..., d \) has a sequence that starts with an element in \( D \) and then is followed by elements in \( \alpha \) and \( \beta \). Every point that is in a preimage of the Cantor set of circles that surrounds the origin that is in the trap door \( T \) has a sequence that starts with a \( \beta_i \) then it is followed by an element in \( D \) and then consists of elements in \( \alpha \) and \( \beta \). We extend the identifications mentioned above to every curve in every Cantor set of circles in the Julia set of \( f_\lambda \).

To extend this definition to all of \( J \), we let \( \Sigma' \) denote the set of sequences involving all the symbols from the sets \( \alpha, \beta \) and \( D \) with no restrictions; that is, we let any symbol be followed by any other. Let \( \Sigma \) denote the space \( \Sigma' \) where we extend the above identifications to any pair of sequences that ends in a pair of sequences identified earlier. We endow \( \Sigma \) with the quotient topology. Then, by construction, the Julia set of \( f_\lambda \) is homeomorphic to \( \Sigma \) and \( f_\lambda | J \) is conjugate to the shift map on \( \Sigma \). This finishes the proof of our main results.

## 4.3 Example

In this section we discuss a particular case when \( n = 3, h = 5, d = 4 \) and \( |a| > 1 \). Then we have

\[
 f_\lambda(z) = z^3 + \frac{\lambda}{z^5(z - a)^4}. 
\]
Notice that the degree of $f_\lambda$ is 12 and so there are 22 critical points counted with multiplicity. Infinity is a critical point of order 2. Thus, there are 20 critical points of $f_\lambda$ that satisfy the equation

$$3c^{12}(c - a)^8 = \lambda(5c^4(c - a)^4 + 4(c - a)^3c^5).$$

As can be seen in this equation, the origin is a critical point of order 4 and the pole $a$ is a critical point of order 3. If we remove these solutions we find

$$3c^8(c - a)^5 = \lambda(5(c - a) + 4c)$$

and we see that, when $\lambda \to 0$, there are 8 critical points that approach 0 (the set $S_0$), and 5 critical points that approach $a$ (the set $S_a$). The critical values corresponding to these “free” critical points are determined by

$$v = f_\lambda(c) = c^3 + \frac{3c^3(c - a)}{5(c - a) + 4c}.$$  

Note that if $c \to 0$ then $v \to 0$ and if $c \to a$ then $v \to a^3$.

The preimage of the disk bounded by $\gamma$ is the annulus $\mathcal{A}$ that contains the 8 critical points in $S_0$ and whose closure lies between $\gamma$ and $\Gamma$. The preimages of this annulus are two annuli and, taking preimages of these annuli, we construct a Cantor set of simple closed curves that surrounds the origin.

There are 4 holes near the pole $a$. These holes are outside $\Gamma$ in $B$ and contain preimages of the Cantor set of circles that surrounds the origin. Since the trap door $T$ is a preimage of the region outside $\Gamma$, it also contains preimages of those holes and so does every preimage of $T$. It follows that each annulus in the Cantor sets of simple closed curves contains preimages of the Cantor set of circles. These new Cantor sets
Figure 4.4: Singular Perturbations in the McMullen Domain. Example.

Top left: the Julia set of $f_\lambda(z) = z^3 + \lambda/(z^5(z - a)^4)$ when the pole $a = 1.2$ and $\lambda = .0001$. Top right: magnification about the pole $a$ and, bottom: magnification of Cantor set of circles inside the trap door $T$. The colored regions are $B$ and its preimages. The different shades represent different escape times.
of simple closed curves are preimages of the one that surrounds the origin and so they cannot contain periodic points. Repelling periodic points are dense in the Julia set and so there must be other components in the Julia set accumulating on each curve of these Cantor sets. These other components of the Julia set are points.

Figure 4.4 shows the Julia set of $f_\lambda(z)$ when $a = 1.2$ and $\lambda = .0001$. 
Bibliography


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Civil Engineering degree, Universidad Nacional de La Plata, Argentina, 1999.

Professional Societies:
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Fall 2003, 2004 and 2005: Research Assistantship in the Department of Mathematics and Statistics at Boston University under the direction of Professor Robert L. Devaney to study the dynamics of singularly perturbed rational maps in complex variables.

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1999-2000: Research Scholarship from the Comision de Investigaciones Cientificas (CIC) de la Provincia de Buenos Aires, Argentina, under the direction of Professor Horacio A. Caruso to study nonlinear dynamical systems by means of numerical experiments.

Awards: Honorable Mention in the SIAM DSWeb Tutorial Contest for Graduate Students, 2005.

Publications:


Recreational Mathematics:

Software:

1. Mathematica notebooks for Iterated Function Systems. April 2005. This work received Honorable Mention in the DSWeb Tutorial Contest for Graduate Students of 2005. In these notebooks the concept of an IFS is explained and many examples of applications are presented. We show how to use IFS’s to study the Julia sets of complex functions, contractive affine maps and random sequences. We also explain how to make animations of the attractors obtained with the IFS’s. The files can be downloaded from: http://www.dynamicalsystems.org/tu/tu/.

2. A First Course in Chaotic Dynamical Systems: Experiments with Mathematica. December 2003. This set of Mathematica notebooks is meant to accompany the book A First Course in Chaotic Dynamical Systems by Robert L. Devaney. These programs are used by students taking Chaotic Dynamical Systems in the Department of Mathematics and Statistics at Boston University and they have also been used at MIT in the course Mathematical Exposition in the spring of 2005 as well as in other universities in the USA. There are 12 Mathematica notebooks, 7 of them were specially designed to solve the Experiments shown in the book, the rest are meant to give tools to the students for understanding different aspects of the dynamics of real and complex functions. The files can be downloaded from: http://math.bu.edu/people/bob/MA471/.

3. Nonlinear Dynamics: A Mathematica Lab Notebook. (With David K. Campbell, Andrew Chameski, Gamailel Lodge, Gary Tam and Tom Tanury). January 2003 through January 2007. A set of Mathematica notebooks that are meant to accompany the book, Nonlinear Science: From Paradigms to Practicalities by David K. Campbell. We developed a CD with more than 80 Mathematica notebooks to study one and two dimensional maps, ordinary differential equations and partial differential equations, with more than 500 pictures and more that 50 exercises with written answers.
Talks, Seminars, Expositions:


8. Tuesday Evening Lectures at the Graduate Painting and Sculpture Department at Boston University. December 12, 2006. Title: Nonlinear Dynamics: Chaos and Fractals.


11. Curso de Especialización Epistemología de la Ciencia y la Tecnología. Dictado por el Dr. Ricardo Gomez. UNLP, Argentina. Agosto y Octubre de 2000 con una duración de 40 horas.


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During the years of 2004 through 2008 I participated as a Teaching Fellow in the course Differential Equations MA 226. I directed the discussion sections for this course and many times I substituted Prof. Robert L. Devaney and gave the lectures in the 150 student course. I also participated in the production work for the 3rd edition of the textbook Differential Equations by Paul Blanchard, Robert L. Devaney and Dick Hall, Thomson Learning, 2006.

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