1.1 Systems of Linear Equations
Section 1.1: Systems of Linear Equations

A linear equation:

\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \]

EXAMPLE:

\[ 4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3 \]

\[ \downarrow \quad \text{rearranged} \quad \downarrow \quad \text{rearranged} \]

\[ 3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6} \]

Not linear:

\[ 4x_1 - 6x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 7 \]

A system of linear equations (or a linear system):

A collection of one or more linear equations involving the same set of variables, say, \( x_1, x_2, \ldots, x_n \).

A solution of a linear system:

A list \((s_1, s_2, \ldots, s_n)\) of numbers that makes each equation in the system true when the values \( s_1, s_2, \ldots, s_n \) are substituted for \( x_1, x_2, \ldots, x_n \), respectively.
EXAMPLE Two equations in two variables:
\[
\begin{align*}
    x_1 + x_2 &= 10 \\
    -x_1 + x_2 &= 0
\end{align*}
\]
\[
\begin{align*}
    x_1 - 2x_2 &= -3 \\
    2x_1 - 4x_2 &= 8
\end{align*}
\]

one unique solution

no solution

\[
\begin{align*}
    x_1 + x_2 &= 3 \\
    -2x_1 - 2x_2 &= -6
\end{align*}
\]

infinitely many solutions

BASIC FACT: A system of linear equations has either
(i) exactly one solution (consistent) or
(ii) infinitely many solutions (consistent) or
(iii) no solution (inconsistent).
EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

i) The planes intersect in one point. \textit{(one solution)}

ii) The planes intersect in one line. \textit{(infinitely many solutions)}

iii) There is not point in common to all three planes. \textit{(no solution)}
The **solution set:**
- The set of all possible solutions of a linear system.

**Equivalent systems:**
- Two linear systems with the same solution set.

**STRATEGY FOR SOLVING A SYSTEM:**
- Replace one system with an equivalent system that is easier to solve.

**EXAMPLE:**

\[
\begin{align*}
    x_1 - 2x_2 & = -1 \\
-x_1 + 3x_2 & = 3 \\
\end{align*}
\]

\[
\begin{align*}
    x_1 & = 2 \\
    x_2 & = 2 \\
\end{align*}
\]
\[ x_1 - 2x_2 = -1 \]
\[ -x_1 + 3x_2 = 3 \]
\[ x_1 - 2x_2 = -1 \]
\[ x_2 = 2 \]

\[ x_1 = 3 \]
\[ x_2 = 2 \]
Matrix Notation

\[
x_1 - 2x_2 = -1 \\
-x_1 + 3x_2 = 3
\]

(coefficient matrix)

\[
x_1 - 2x_2 = -1 \\
x_2 = 2
\]

(augmented matrix)
Elementary Row Operations:
1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

**Row equivalent matrices:** Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

**Fact about Row Equivalence:** If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
EXAMPLE:

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{bmatrix}
\]
\[
\begin{align*}
x_1 &= 29 \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\
x_2 &= 16 \quad & 0 & 1 & 0 & 16 \\
x_3 &= 3 \quad & 0 & 0 & 1 & 3 
\end{bmatrix} \\
\end{align*}
\]

**Solution:** \((29, 16, 3)\)

**Check:** Is \((29, 16, 3)\) a solution of the original system?

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{align*}
\]

\[
\begin{align*}
(29) - 2(16) + 3 &= 29 - 32 + 3 = 0 \\
2(16) - 8(3) &= 32 - 24 = 8 \\
-4(29) + 5(16) + 9(3) &= -116 + 80 + 27 = -9
\end{align*}
\]
Two Fundamental Questions (Existence and Uniqueness)

1) Is the system consistent; (i.e. does a solution exist?)

2) If a solution exists, is it unique? (i.e. is there one & only one solution?)

**EXAMPLE:** Is this system consistent?

\[
\begin{align*}
 x_1 & - 2x_2 + x_3 = 0 \\
 2x_2 & - 8x_3 = 8 \\
 -4x_1 & + 5x_2 + 9x_3 = -9
\end{align*}
\]

In the last example, this system was reduced to the triangular form:

\[
\begin{align*}
 x_1 & - 2x_2 + x_3 = 0 \\
 x_2 & - 4x_3 = 4 \\
 x_3 & = 3
\end{align*}
\]

\[
\begin{bmatrix}
 1 & -2 & 1 & 0 \\
 0 & 1 & -4 & 4 \\
 0 & 0 & 1 & 3
\end{bmatrix}
\]

This is sufficient to see that the system is consistent and unique. Why?
EXAMPLE: Is this system consistent?

\[
\begin{align*}
3x_2 - 6x_3 &= 8 \\
x_1 - 2x_2 + 3x_3 &= -1 \\
5x_1 - 7x_2 + 9x_3 &= 0
\end{align*}
\]

\[
\begin{bmatrix}
0 & 3 & -6 & 8 \\
1 & -2 & 3 & -1 \\
5 & -7 & 9 & 0
\end{bmatrix}
\]
**EXAMPLE:** For what values of $h$ will the following system be consistent?

\[
\begin{align*}
3x_1 - 9x_2 &= 4 \\
-2x_1 + 6x_2 &= h
\end{align*}
\]
1.1: 3,5,8,9,12,13,17,18,19,22

Note the practice problems at the end of each chapter