Section 2.2

The Derivative Function
Given function $f$, we can

- estimate $f'$, the derivative at a point, and
- sketch $f'$ as a function

Consider $f$ given by the table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0</td>
<td>10</td>
<td>30</td>
<td>40</td>
<td>45</td>
<td>60</td>
<td>80</td>
<td>90</td>
<td>95</td>
<td>95</td>
<td>95</td>
</tr>
</tbody>
</table>

Estimate a table of values for $f'$. 
Figure 2.19

Table 2.4  Estimated values of derivative of function in Figure 2.19

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivative at $x$</td>
<td>$6$</td>
<td>$2$</td>
<td>$-1$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

For a function $f$, we define the **derivative function**, $f'$, by

$$f'(x) = \text{Instantaneous rate of change of } f \text{ at } x.$$
The graph of $f$ is in Figure 2.22. Which of the graphs (a)–(c) is a graph of the derivative, $f'$?

Solution

Since the graph of $f(x)$ is horizontal at $x = -1$ and $x = 2$, the derivative is zero there. Therefore, the graph of $f'(x)$ has $x$-intercepts at $x = -1$ and $x = 2$.

The function $f$ is decreasing for $x < -1$, increasing for $-1 < x < 2$, and decreasing for $x > 2$. The derivative is positive (its graph is above the $x$-axis) where $f$ is increasing, and the derivative is negative (its graph is below the $x$-axis) where $f$ is decreasing. The correct graph is (c).

Example 3
If $f' > 0$ on an interval, then $f$ is increasing over that interval.
If $f' < 0$ on an interval, then $f$ is decreasing over that interval.
If $f' = 0$ on an interval, then $f$ is constant over that interval.
Section 2.3

Interpretations of the Derivative
The derivative of $f(x)$ at $x = a$ is

the slope of the line tangent to $f$ at $x = a$
and

the instantaneous rate of change in $f$ at $x = a$
and

is approximated by the average change in $f$ over a small interval containing $a$
and

has the same units as $\frac{\Delta f}{\Delta x}$ or $\frac{\Delta y}{\Delta x}$. 
\[ f'(x) = \frac{dy}{dx}. \]

- The units of the derivative of a function are the units of the dependent variable divided by the units of the independent variable.
- If the derivative of a function is not changing rapidly near a point, then the derivative is approximately equal to the change in the function when the independent variable increases by 1 unit.
If \( s = f(t) \) gives \( s \) meters at \( t \) hours, then \( s'(t) \) measures ...

Suppose \( s(5) = 10 \) and \( s'(5) = 2 \). What does this mean?

Estimate \( s(6), s(7), s(8), s(4) \)
If \( v = f(t) \) gives velocity in meters per second at \( t \) seconds, then \( v'(t) \) measures...

Suppose \( v(10) = 20 \) and \( v'(10) = 3 \). What does this mean?

Estimate \( v(11) \), \( v(12) \), \( v(9) \).
Local Linear Approximation

\[ \Delta y \approx f'(x) \Delta x \quad \text{for } \Delta x \text{ near 0.} \]

Example 3

Climbing health care costs have been a source of concern for some time. Use the data\(^6\) in Table 2.7 to estimate average (per consumer unit) expenditures in 2005 and 2020.

**Table 2.7**  
*Average yearly health care costs (per consumer unit) for various years since 1990*

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Per capita expenditure ($)</td>
<td>1480</td>
<td>1732</td>
<td>1903</td>
<td>2066</td>
<td>2350</td>
</tr>
</tbody>
</table>

**Solution**

Health care costs increased throughout the period shown. Between 2000 and 2002, they increased \((2350 - 2066)/2 = \$142\) per year. To make estimates beyond 2002 we assume that costs continue to climb at the same rate. Therefore, we estimate

\[
\text{Costs in 2005} = \text{Costs in 2002} + \text{Change in costs}
\]
\[
\approx \$2350 + \$142 \cdot 3 = \$2776.
\]

Since 2020 is 18 years beyond 2002,

\[
\text{Costs in 2020} \approx \$2350 + \$142 \cdot 18 = \$4906.
\]
In general, given $f(a)$ and $f'(a)$,

$$f(x) \approx f(a) + \Delta x f'(a)$$

or

$$f(x) \approx f(a) + (x - a)f'(a)$$