Section 9.1

Understanding Functions of Two Variables
Table 9.1  Revenue from ticket sales as a function of $x$ and $y$

<table>
<thead>
<tr>
<th>Number of discount tickets, $y$</th>
<th>Number of full price tickets, $x$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
</tr>
<tr>
<td>200</td>
<td>75,000</td>
</tr>
<tr>
<td>400</td>
<td>115,000</td>
</tr>
<tr>
<td>600</td>
<td>155,000</td>
</tr>
<tr>
<td>800</td>
<td>195,000</td>
</tr>
<tr>
<td>1000</td>
<td>235,000</td>
</tr>
</tbody>
</table>

$R = 350x + 200y$. 

*Applied Calculus, 3/E* by Deborah Hughes-Hallet
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$C(x, y) = x^2 - 3y$

- Use a table to investigate this function for $x = 0, 1, 2, 3$ and $y = 0, 1, 2, 3$.
- Use your calculator to sketch this function assuming $y$ is a constant.
- Use your calculator to sketch this function assuming $x$ is a constant.
Concentration of a Drug in the Blood

When a drug is injected into muscle tissue, it diffuses into the bloodstream. The concentration of the drug in the blood increases until it reaches a maximum, and then decreases. The concentration, \( C \) (in mg per liter), of the drug in the blood is a function of two variables: \( x \), the amount (in mg) of the drug given in the injection, and \( t \), the time (in hours) since the injection was administered. We are told that

\[
C = f(x, t) = te^{-t(5-x)} \quad \text{for } 0 \leq x \leq 4 \text{ and } t \geq 0.
\]

Example 3

In terms of the drug concentration in the blood, explain the significance of the cross-sections:

(a) \( f(4, t) \)  
(b) \( f(x, 1) \)
5. The balance, $B$, in dollars, in a bank account depends on the amount deposited, $A$ dollars, the annual interest rate, $r\%$, and the time, $t$, in months since the deposit, so $B = f(A, r, t)$.

(a) Is $f$ an increasing or decreasing function of $A$? Of $r$? Of $t$?
(b) Interpret the statement $f(1250, 1, 25) \approx 1276$. Give units.

Problem 5
Section 9.2

Contour Diagrams
1. Figure 9.20 shows contour diagrams of temperature in °C in a room at three different times. Describe the heat flow in the room. What could be causing this?
7. Figure 9.25 is a contour diagram of the monthly payment on a 5-year car loan as a function of the interest rate and the amount you borrow. The interest rate is 13% and you borrow $6000.

(a) What is your monthly payment?
(b) If interest rates drop to 11%, how much more can you borrow without increasing your monthly payment?
(c) Make a table of how much you can borrow, without increasing your monthly payment, as a function of the interest rate.

Problem 7
27. Match tables (a)–(d) with the contour diagrams (I)–(IV) in Figure 9.31.

Problem 27 (a) (Contour diagrams on next slide)
Problem 27 (b) (continued)
An antibiotic is given to a patient. The percentage of the dose excreted, $P$, is a function of glomerular filtration rate (GFR), in ml/min, and time, in hours.

1. If a patient has GFR = 30 ml/min, how long will it take to excrete 20% of a dose?
2. If the patient has GFR = 30 ml/min, how long will it take?
3. Is $P$ an increasing or decreasing function of time?
4. Is $P$ an increasing or decreasing function of GFR?
Section 9.3

Partial Derivatives
Partial Derivatives of $f$ With Respect to $x$ and $y$

The **partial derivative of $f$ with respect to $x$** at $(a, b)$ is the derivative of $f$ with $y$ constant:

$$f_x(a, b) = \text{Rate of change of } f \text{ with } y \text{ fixed at } b, \text{ at the point } (a, b) = \lim_{{h \to 0}} \frac{f(a + h, b) - f(a, b)}{h}.$$ 

The **partial derivative of $f$ with respect to $y$** at $(a, b)$ is the derivative of $f$ with $x$ constant:

$$f_y(a, b) = \text{Rate of change of } f \text{ with } x \text{ fixed at } a, \text{ at the point } (a, b) = \lim_{{h \to 0}} \frac{f(a, b + h) - f(a, b)}{h}.$$ 

If we think of $a$ and $b$ as variables, $a = x$ and $b = y$, we have the **partial derivative functions** $f_x(x, y)$ and $f_y(x, y)$.

**Alternative Notation for Partial Derivatives**

If $z = f(x, y)$ we can write

$$f_x(x, y) = \frac{\partial z}{\partial x} \quad \text{and} \quad f_y(x, y) = \frac{\partial z}{\partial y}$$

$$f_x(a, b) = \left. \frac{\partial z}{\partial x} \right|_{(a, b)} \quad \text{and} \quad f_y(a, b) = \left. \frac{\partial z}{\partial y} \right|_{(a, b)}$$
Table 9.3.1 gives $H =$ temperature in °F at time $t$ minutes since noon on a certain day. If $H = f(t)$, estimate $f'(10)$. What are the units of this derivative? What is its meaning?

<table>
<thead>
<tr>
<th>$t$ (min)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$ (°F)</td>
<td>60</td>
<td>65</td>
<td>68</td>
<td>70</td>
</tr>
</tbody>
</table>

Table 9.3.2 gives values of $H = g(x, t)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100$</td>
<td>72</td>
<td>75</td>
<td>80</td>
<td>84</td>
</tr>
<tr>
<td>$200$</td>
<td>67</td>
<td>72</td>
<td>76</td>
<td>80</td>
</tr>
<tr>
<td>$300$</td>
<td>60</td>
<td>65</td>
<td>68</td>
<td>70</td>
</tr>
<tr>
<td>$400$</td>
<td>53</td>
<td>59</td>
<td>64</td>
<td>66</td>
</tr>
</tbody>
</table>
9. The sales of a product, \( S = f(p, a) \), is a function of the price, \( p \), of the product (in dollars per unit) and the amount, \( a \), spent on advertising (in thousands of dollars).

(a) Do you expect \( f_p \) to be positive or negative? Why?
(b) Explain the meaning of the statement \( f_a(8, 12) = 150 \) in terms of sales.

Problem 9
11. Figure 9.42 shows a contour diagram for the monthly payment $P$ as a function of the interest rate, $r\%$, and the amount, $L$, of a 5-year loan. Estimate $\partial P/\partial r$ and $\partial P/\partial L$ at the point where $r = 8$ and $L = 5000$. Give the units and the financial meaning of your answers.
Change in $f$ \approx \text{Rate of change in } x\text{-direction} \cdot \Delta x + \text{Rate of change in } y\text{-direction} \cdot \Delta y

\[ \Delta f \approx f_x \cdot \Delta x + f_y \cdot \Delta y \]

17. For a function $f(r, s)$, we are given $f(50, 100) = 5.67$, and $f_r(50, 100) = 0.60$, and $f_s(50, 100) = -0.15$. Estimate $f(52, 108)$. 
Section 9.4

Computing Partial Derivatives Algebraically
3. $f_x$ and $f_y$ if $f(x, y) = 2x^2 + 3y^2$

5. $\frac{\partial P}{\partial r}$ if $P = 100e^{rt}$

9. $z_x$ if $z = x^2y + 2x^5y$
The Second-Order Partial Derivatives of $z = f(x, y)$

\[
\frac{\partial^2 z}{\partial x^2} = f_{xx} = (f_x)_x, \quad \frac{\partial^2 z}{\partial x \partial y} = f_{yx} = (f_y)_x, \\
\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = (f_x)_y, \quad \frac{\partial^2 z}{\partial y^2} = f_{yy} = (f_y)_y.
\]
If $f_{xy}$ and $f_{yx}$ are continuous at $(a, b)$, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$
Section 9.5

Critical Points and Optimization
● $f$ has a **local maximum** at $P_0$ if $f(P_0) \geq f(P)$ for all points $P$ near $P_0$
• \( f \) has a **local maximum** at \( P_0 \) if \( f(P_0) \geq f(P) \) for all points \( P \) near \( P_0 \)

• \( f \) has a **local minimum** at \( P_0 \) if \( f(P_0) \leq f(P) \) for all points \( P \) near \( P_0 \)
- $f$ has a **local maximum** at $P_0$ if $f(P_0) \geq f(P)$ for all points $P$ near $P_0$
- $f$ has a **local minimum** at $P_0$ if $f(P_0) \leq f(P)$ for all points $P$ near $P_0$
- $f$ has a **global maximum** at $P_0$ if $f(P_0) \geq f(P)$ for all points $P$ in $\mathbb{R}$
• $f$ has a **local maximum** at $P_0$ if $f(P_0) \geq f(P)$ for all points $P$ near $P_0$
• $f$ has a **local minimum** at $P_0$ if $f(P_0) \leq f(P)$ for all points $P$ near $P_0$
• $f$ has a **global maximum** at $P_0$ if $f(P_0) \geq f(P)$ for all points $P$ in $\mathbb{R}$
• $f$ has a **global minimum** at $P_0$ if $f(P_0) \leq f(P)$ for all points $P$ in $\mathbb{R}$
Table 9.9 gives a table of values for a function $f(x, y)$. Estimate the location and value of any global maxima or minima for $0 \leq x \leq 1$ and $0 \leq y \leq 20$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>80</td>
<td>86</td>
<td>91</td>
<td>87</td>
<td>82</td>
</tr>
<tr>
<td>0.2</td>
<td>84</td>
<td>90</td>
<td>95</td>
<td>91</td>
<td>86</td>
</tr>
<tr>
<td>0.4</td>
<td>82</td>
<td>88</td>
<td>93</td>
<td>89</td>
<td>84</td>
</tr>
<tr>
<td>0.6</td>
<td>76</td>
<td>73</td>
<td>88</td>
<td>84</td>
<td>79</td>
</tr>
<tr>
<td>0.8</td>
<td>71</td>
<td>77</td>
<td>77</td>
<td>78</td>
<td>73</td>
</tr>
<tr>
<td>1.0</td>
<td>65</td>
<td>71</td>
<td>71</td>
<td>72</td>
<td>67</td>
</tr>
</tbody>
</table>

**Solution**

The global maximum value of the function appears to be 95 at the point $(0.2, 10)$. Since the table only gives certain values, we cannot be sure that this is exactly the maximum. (The function might have a larger value at, for example, $(0.3, 11)$.) The global minimum value of this function on the points given is 65 at the point $(1, 0)$.

**Example 1**
Figure 9.49 gives a contour diagram for a function \( f(x, y) \). Estimate the location and value of any local maxima or minima. Are any of these global maxima or minima on the square shown?

Solution

There is a local maximum of above 8 near the point \((6, 5)\), a local maximum of above 6 near the point \((2, 6)\), and a local minimum of below 3 near the point \((3, 2)\). The value above 8 is the global maximum and the value below 3 is the global minimum on the given domain.

Example 2
If a function $f(x, y)$ has a local maximum or minimum at a point $(x_0, y_0)$ not on the boundary of the domain of $f$, then either

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

or (at least) one partial derivative is undefined at the point $(x_0, y_0)$. Points where each of the partial derivatives is either zero or undefined are called **critical points**.

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**Example 3**

Find and analyze the critical points of $f(x, y) = x^2 - 2x + y^2 - 4y + 5$.

**Solution**

To find the critical points, we set both partial derivatives equal to zero:

$$f_x(x, y) = 2x - 2 = 0,$$
$$f_y(x, y) = 2y - 4 = 0.$$

Solving these equations gives $x = 1$ and $y = 2$. Hence, $f$ has only one critical point, namely $(1, 2)$. What is the behavior of $f$ near $(1, 2)$? The values of the function in Table 9.10 suggest that the function has a local minimum value of 0 at the point $(1, 2)$.

**Click for Table 9.10**
Table 9.10 \quad \textit{Values of } f(x, y) \textit{ near the point } (1, 2) \\

<table>
<thead>
<tr>
<th></th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>0.08</td>
<td>0.05</td>
<td>0.04</td>
<td>0.05</td>
<td>0.08</td>
</tr>
<tr>
<td>1.9</td>
<td>0.05</td>
<td>0.02</td>
<td>0.01</td>
<td>0.02</td>
<td>0.05</td>
</tr>
<tr>
<td>2.0</td>
<td>0.04</td>
<td>0.01</td>
<td>0.00</td>
<td>0.01</td>
<td>0.04</td>
</tr>
<tr>
<td>2.1</td>
<td>0.05</td>
<td>0.02</td>
<td>0.01</td>
<td>0.02</td>
<td>0.05</td>
</tr>
<tr>
<td>2.2</td>
<td>0.08</td>
<td>0.05</td>
<td>0.04</td>
<td>0.05</td>
<td>0.08</td>
</tr>
</tbody>
</table>
Second Derivative Test for Functions of Two Variables
Suppose \((x_0, y_0)\) is a critical point where \(f_x(x_0, y_0) = f_y(x_0, y_0) = 0\). Let

\[
D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.
\]

- If \(D > 0\) and \(f_{xx}(x_0, y_0) > 0\), then \(f\) has a local minimum at \((x_0, y_0)\).
- If \(D > 0\) and \(f_{xx}(x_0, y_0) < 0\), then \(f\) has a local maximum at \((x_0, y_0)\).
- If \(D < 0\), then \(f\) has neither a local maximum or minimum at \((x_0, y_0)\).
- If \(D = 0\), the test is inconclusive.

5. \(f(x, y) = x^2 + xy + 3y\)